- Particle – A Particle is defined as an object whose mass is finite but whose size and internal structure can be neglected.

- System – A system is a collection of a very large number of particles which mutually interact with one-another.

- Internal force – The mutual forces exerted by the particles of a system on one-another are called internal forces.

- External forces – The outside force exerted on an object by external agency is called an external force.

- Center of mass – The center of mass of a system of particles is that single point which move in the same way in which a single particle having the total mass of the system and acted upon by the same external force would move.

or.

The center of mass of a body is a point where the whole mass of the body is supposed to be concentrated for describing its translatory motion.

- Center of gravity – It is a point at which the resultant of the gravitational forces on all the particles of the body acts or a point where whole weight may be assumed to act.

- Derivation of expression for the center of mass of a two particle system.

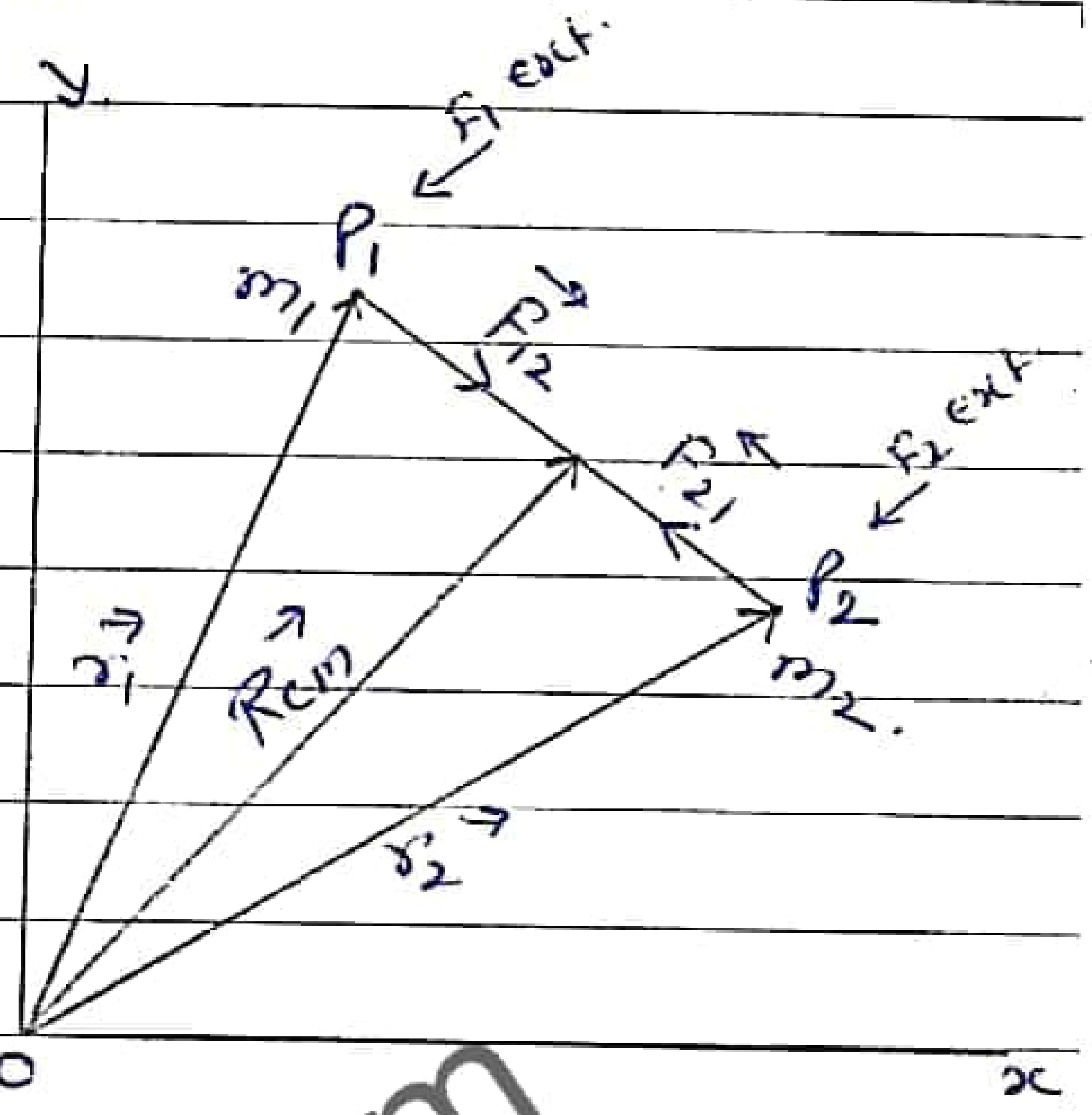
Consider a system of two particles P_1 and P_2 of masses m_1 and m_2. Let $\vec{r_1}$ and $\vec{r_2}$ be their position vector at any

instant t with respect to origin O
the velocity and acceleration vectors of the two Particles are.

$\vec{v_1} = \frac{d\vec{r_1}}{dt}$; $\vec{v_2} = \frac{d\vec{r_2}}{dt}$.

$\vec{a_1} = \frac{d\vec{v_1}}{dt} = \frac{d^2\vec{r_1}}{dt^2}$

$a_2 = \frac{d\vec{v_2}}{dt} = \frac{d^2\vec{r_2}}{dt^2}$



Total force $\vec{F_1}$ acting on Particle P_1 is the sum of internal force $\vec{F_{12}}$ due to P_2 and external force $\vec{F_1}_{ext}$. so $\vec{F_1} = \vec{F_{12}} + \vec{F_1}_{ext}$
$\vec{F_2} = F_{21} + F_{2\,ext}$.

$$\vec{F_1} = m_1\vec{a_1} = \vec{F_{12}} + \vec{F_1}_{ext}.$$
$$\vec{F_2} = m_2\vec{a_2} = \vec{F_{21}} + \vec{F_2}_{ext}$$

$F_{12} = -F_{21}$ By the Newton's third law.

so $F_{12} + F_{21} = 0$.

$\vec{F} = F_{1\,ext} + F_{2\,ext}$ is the total external force acting on the system

Suppose the total mass of the system of two Particle is M.

then $M = m_1 + m_2$.

the total external force F acting on the system of mass M Produce acceleration a_{cm} then by Newton's second law.

$$Ma_{cm} = F.$$

then $F_{ex.} = F_{1\,ex} + F_{2\,ex.}$

$$Ma_{cm} = m_1a_1 + m_2a_2.$$

$$M.\frac{d^2R_{cm}}{dt^2} = m_1\frac{d^2r_1}{dt^2} + m_2\frac{d^2r_2}{dt^2}.$$

$$\text{then } \frac{d^2}{dt^2}R_{cm} = \frac{d^2}{dt^2}\left(\frac{m_1r_1 + m_2r_2}{m_1 + m_2}\right)$$

$$\text{then } R_{cm} = \frac{m_1r_1 + m_2r_2}{m_1 + m_2}.$$

If there is a system of n Particles having masses $m_1, m_2, \cdots m_n$ and Position vectors $r_1, r_2, \cdots r_n$ then.

$$R_{cm} = \frac{m_1 r_1 + m_2 r_2 + \cdots + m_n r_n}{m_1 + m_2 + \cdots + m_n}.$$

Cartesian Co-ordinates of the Center of mass.

$$x_{cm} = \frac{m_1 x_1 + m_2 x_2 + \cdots + m_n x_n}{m_1 + m_2 + \cdots + m_n} = \frac{\Sigma m x}{M}.$$

$$y_{cm} = \frac{m_1 y_1 + m_2 y_2 + \cdots + m_n y_n}{m_1 + m_2 + \cdots + m_n} = \frac{\Sigma m y}{M}$$

$$z_{cm} = \frac{m_1 z_1 + m_2 z_2 + \cdots + m_n z_n}{m_1 + m_2 + \cdots + m_n} = \frac{\Sigma m z}{M}.$$

Q. Show that in the absence of any external force the velocity of the center of mass remains constant.

Ans. Suppose an external force F acts on a system of mass M and produce an acceleration $\vec{a}_{cm}$ in its center of mass then $F = M a_{cm}$.

If $F = 0$ then $M a_{cm} = 0$ or $a_{cm} = 0$.

$$\frac{dv_{cm}}{dt} = 0.$$

Hence in the absence of any external force, the center of mass of a system moves with a uniform velocity.

Q. Show that the total linear momentum of a system of particles is conserved in the absence of any external force and also show the total linear momentum of the system is equal to the product of the total mass of the system and the velocity of the center of mass.

Ans. Let a system of n particles of masses $m_1, m_2 \cdots m_n$, suppose the forces $F_1, F_2 \cdots F_n$ exerted on them produce acceleration $a_1, a_2, \cdots a_n$

In the absence of any external force

$$F_{tot} = 0 = F_1 + F_2 + \cdots + F_n.$$

or $\quad m_1 a_1 + m_2 a_2 + \cdots + m_n a_n = 0.$

or $\quad m_1 \frac{dv_1}{dt} + m_2 \frac{dv_2}{dt} + \cdots + m_n \frac{dv_n}{dt} = 0.$

or $\quad \frac{d}{dt}(m_1 v_1 + m_2 v_2 + \cdots + m_n v_n) = 0.$

or $\quad m_1 v_1 + m_2 v_2 + \cdots + m_n v_n = \text{Constant}.$

or $\quad P_1 + P_2 + \cdots + P_n = P = \text{Constant}.$

where P is the total linear momentum of the system.

Now the Position vector of the center of mass.

$$R_{cm} = \frac{m_1 r_1 + m_2 r_2 + \cdots + m_n r_n}{m_1 + m_2 + \cdots + m_n}$$

$$R_{cm} = \frac{1}{M}[m_1 r_1 + m_2 r_2 + \cdots + m_n r_n]$$

Differentiating both side w.r.t time t:-

$$\frac{d}{dt} R_{cm} = \frac{1}{M}\left[m_1 \frac{d}{dt} r_1 + m_2 \frac{d}{dt} r_2 + \cdots + m_n \frac{d}{dt} r_n\right]$$

$$V_{cm} = \frac{1}{M}[m_1 v_1 + m_2 v_2 + \cdots + m_n v_n]$$ velocity of center of mass.

$$= \frac{1}{M}[P_1 + P_2 + \cdots + P_n].$$

$$M V_{cm} = P_1 + P_2 + \cdots + P_n = P$$

$$[P = P_1 + P_2 + \cdots + P_n]$$

so $P = M V_{cm}$

So the total linear momentum of a system of Particles is equal to the Product of the total mass of the system and the velocity of its center of mass.

- Binary Star — Two stars bound to each other by the gravitational force and orbiting around their common center of mass are called binary stars.

Rigid body – A body is said to be rigid if it does not undergo any change in its size and shape, however large the external force may be acting on it.

– Center of mass of a rigid body – It is a point at a fixed position with respect to the body as a whole.

The position of center of mass of a rigid body depends upon two factors (i) The geometrical shape (ii) The distribution of mass of the body

For bodies having regular geometrical shape and uniform mass distribution; the center of mass lies at their geometrical center.

– Rotational motion of a rigid body – A body is said to possess rotational motion if all its particles move along circles in parallel lines plane. The centers of these circles lie on a fixed line perpendicular to the parallel plane and is called the axis of rotation. and all the particles have the same angular velocity.

– Derivation of three equations of rotational motion under constant angular acceleration.

a. Derivation of first law. – Consider a rigid body rotating about a fixed axis with constant angular acceleration α.

then $\alpha = \frac{d\omega}{dt}$ or $d\omega = \alpha . dt$. —(i)

Let At time $t=0$ angular velocity $\omega = \omega_0$

At time $t=t$. angular velocity $\omega = \omega$

Integrating the eq. (i) within these limits.

$$\int_{\omega_0}^{\omega} d\omega = \alpha \int_0^t dt.$$

$$[\omega]_{\omega_0}^{\omega} = \alpha [t]_0^t$$

$$\omega - \omega_0 = \alpha t.$$

$$\omega = \omega_0 + \alpha t \qquad — (2)$$

- Second equation – Let ω be the angular velocity of a rigid body at any instant t.

$$\omega = \frac{d\theta}{dt} \quad \text{or.} \quad d\theta = \omega . dt \qquad — (3).$$

At $t = 0$ $\quad \theta = 0.$

$t = t$ $\quad \theta = \theta.$

Intergrating eq. (3) within these limits.

$$\int_0^\theta d\theta = \omega \int_0^t dt = \int_0^t (\omega_0 + \alpha t)\, dt.$$

$$= \int_0^t \omega_0 dt + \int_0^t \alpha t . dt$$

$$[\theta]_0^\theta = \omega_0 (t)_0^t + \alpha . \left[\frac{t^2}{2}\right]_0^t.$$

$$\theta = \omega_0 t + \frac{1}{2}\alpha t^2. \qquad — (4)$$

- third equation – the angular acceleration $\alpha = \frac{d\omega}{dt}.$

then $\alpha = \frac{d\omega}{d\theta} \times \frac{d\theta}{dt}. = \frac{d\omega}{d\theta} . \omega.$

$$\text{or.} \quad \omega . d\omega = \alpha . d\theta. \qquad — (5).$$

At $t = 0$ $\quad \theta = 0$ and $\omega = \omega_0.$

$t = t$ $\quad \theta = \theta$ and $\omega = \omega.$

Integrating eq. (5) within these limits.

$$\int_{\omega_0}^{\omega} \omega . d\omega = \alpha \int_0^\theta d\theta.$$

$$\left[\frac{\omega^2}{2}\right]_{\omega_0}^{\omega} = \alpha [\theta]_0^\theta.$$

$$\frac{\omega^2}{2} - \frac{\omega_0^2}{2} = \alpha\theta \quad \text{or} \quad \omega^2 - \omega_0^2 = 2\alpha\theta. \qquad — (6).$$

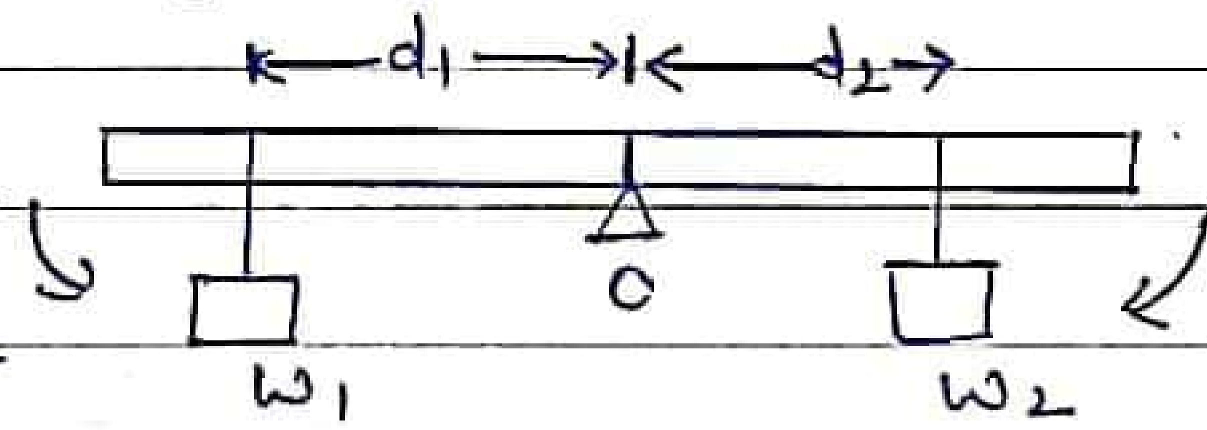
Torque or moment of force – the torque or moment of force is the turning effect of the force about the axis of rotation. It is measured as the product of the magnitude of the force and the perpendiculer distance between the line of action of the force and the axis of rotation.

$$\tau = F \times d.$$

The S.I. unit of torque is N.meter and its dimensional formula is. ML^2T^{-2}

The moment of force is taken positive if the turning tendency of the force is anticlockwise and negative if it is clockwise.

– Principle of moment – When a body is in rotational equilibrium, the sum of the clockwise moments about any point is equal to the sum of the anticlockwise moments about that point or the algebraic sum of moments about any point is zero –



Let a uniform rod free to rotate on a pivot O. Two weight W_1 and W_2 are hung from it at distance d_1 and d_2 from the pivot O

then clockwise moment about O $= F_1 \times d_1 = W_1 \times d_1$

Anticlockwise moment $= F_2 \times d_2 = W_2 \times d_2$.

then According to the principle of moments, the rod will be horizonal or in rotational equilibrium.

$$F_1 \times d_1 = F_2 \times d_2.$$

$$W_1 \times d_1 = W_2 \times d_2.$$

this is sometimes called the lever principle.

– Couple – A pair of equal and opposite forces acting on a body along two different lines of action constituate a couple.

The moment of a couple is equal to the product of either of the forces and the perpendiculer distance called the arm of the Couple between their line of action.

moment of a Couple is independent of the choice of the fulcrum or the Point of rotation.

$$\text{Torque } \tau = F \times d.$$
$$= Fd \sin\theta.$$

the direction of the torque is Perpendicular to the Plane Containing d and F and determined by right hand rule.

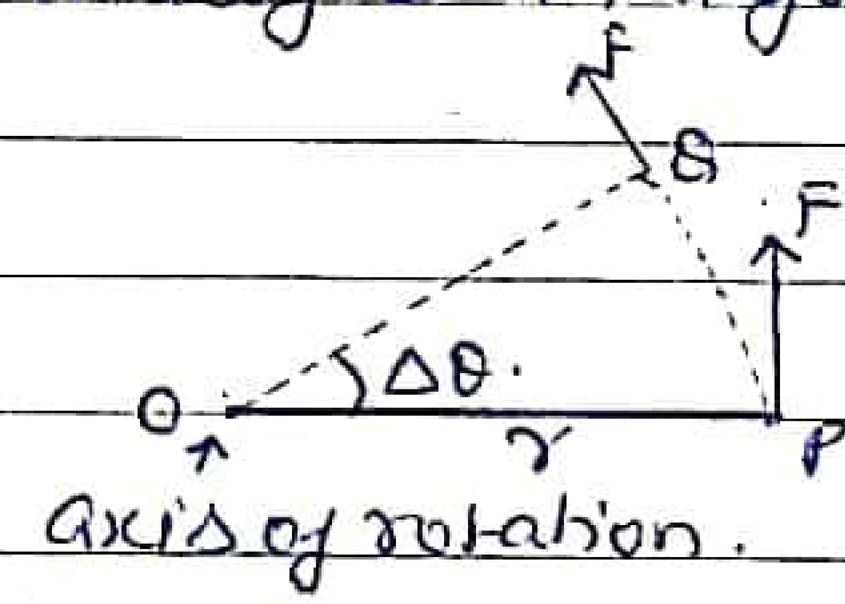
Special Case. (i) when $\theta = 180°$ then $\sin 180 = 0$.

$$\text{so } \tau = 0$$

(ii) when $\theta = 90°$ then $\sin 90° = 1$

so $\tau = Fd$ and it is maximum.

— work done by a torque — Supposed a body undergoes an angular displacement $\Delta\theta$ under the action of a tangential force F.



The work done in the rotational motion of the body or the work done by the torque is.

$$\Delta W = F \times \text{distance along the arc } PQ.$$

But $\Delta\theta = \frac{\text{Arc } PQ}{r}$ so $PQ = r \cdot \Delta\theta$.

then $\Delta W = F \times r\Delta\theta = \tau\Delta\theta$.

In Case of torque applied is not Constant, but variable, the total work done by the torque is.

$$W = \int_{\theta_1}^{\theta_2} \tau \, d\theta$$

Power delivered by a torque —

$$\Delta W = \tau \Delta\theta.$$

dividing both side by Δt.

$$\frac{\Delta W}{\Delta t} = \tau \frac{\Delta\theta}{\Delta t}.$$

$$P = \tau \times \omega$$

— Rectangular Components of torque — For three dimensional motion, the Position vector $\vec{r}$ and torque τ are written as

$$\vec{r} = x\hat{i} + y\hat{j} + z\hat{k} \quad \text{and} \quad \tau = \tau_x\hat{i} + \tau_y\hat{j} + \tau_z\hat{k}$$

$$\vec{F} = F_x\hat{i} + F_y\hat{j} + F_z\hat{k}$$

then

$$\vec{\tau} = \vec{r} \times \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ x & y & z \\ F_x & F_y & F_z \end{vmatrix}$$

$$\tau_x\hat{i} + \tau_y\hat{j} + \tau_z\hat{k} = \hat{i}(yF_z - zF_y) + \hat{j}(zF_x - xF_z) + \hat{k}(xF_y - yF_x)$$

so

$$\tau_x = yF_z - zF_y \; ; \; \tau_y = zF_x - xF_z \; ; \; \tau_z = xF_y - yF_x .$$

— Angular momentum — The angular momentum of a Particle rotating about an axis is defined as the moment of the linear momentum of the Particle about that axis.

S.I. unit of angular momentum is $kg\, m^2/sec$ and its dimensional formula is ML^2T^{-1}.

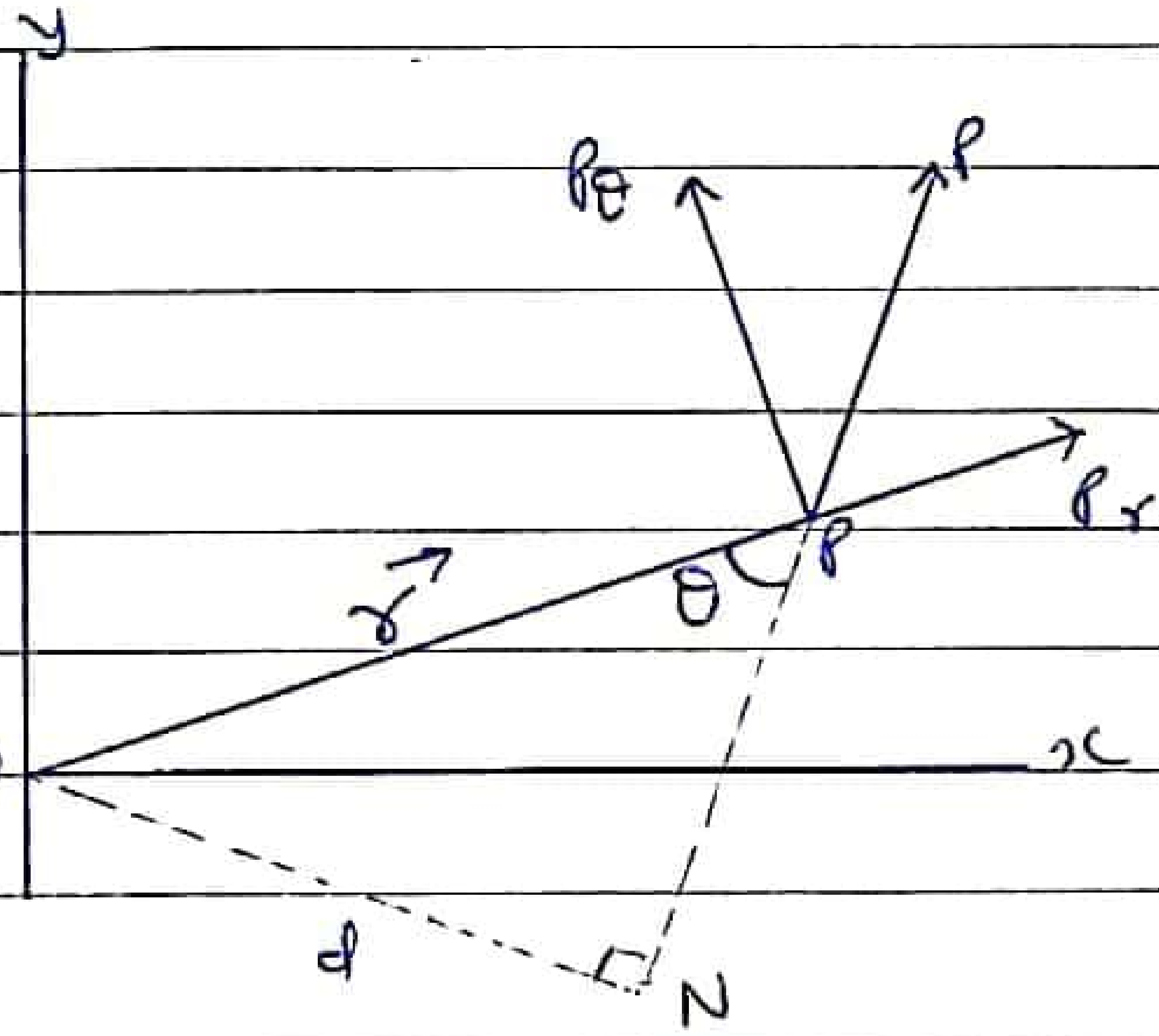
Q. Show that the angular momentum of a Particle is the Product of its linear momentum and the moment arm. also Show that the angular momentum is Produced only by the angular Component of linear momentum.

An. Consider a particle P of mass m whose Position vector relative to the origion O is r, suppose the momentum vector $\vec{P}$ make angle θ with the Position vector $\vec{r}$,

Draw ON Perpendicular to the line of action of linear momentum P, from right angle Δ ONP

$$\frac{ON}{OP} = \frac{d}{r} = \sin\theta .$$

or $d = r\sin\theta$.



this is the Perpendicular distance of the linear momentum from the Point of rotation O and is Called moment arm of the momentum.

$$L = rP\sin\theta = P(r\sin\theta) = Pd.$$

and the Component of linear momentum are.

$P_r = P\cos\theta$ and $P_\theta = P\sin\theta$.

$$\text{so } L = (P\sin\theta)r = P_\theta \times r.$$

Angular momentum = Angular Component of linear momentum × distance from Axis of rotation.

- Angular momentum in term of ractangular Components.

For motion in three dimensions,

$r = x\hat{i} + y\hat{j} + z\hat{k}$ and $P = P_x\hat{i} + P_y\hat{j} + P_z\hat{k}$.

Angular momentum is the cross Product of r and P.

$$\text{so } L = r \times P.$$

$$L_x i + L_y j + L_z k = (xi + yj + zk) \times (P_x i + P_y j + P_z k)$$

$$L_x i + L_y j + L_z k = i(yP_z - zP_y) + j(zP_x - xP_z) + k(xP_y - yP_x)$$

then:

$L_x = yP_z - zP_y$; $L_y = zP_x - xP_z$; $L_z = xP_y - yP_x$.

- Relation between torque and angular momentum.

Torque $\tau = r \times F$

Angular momentum $L = r \times P$.

$$\text{then } \frac{dL}{dt} = r.\frac{dP}{dt} + P.\frac{dr}{dt}.$$

$$= r \times F + P \times v \qquad \because v \times P = v \times mv = 0.$$

$$= r \times F$$

$$\frac{dL}{dt} = \tau.$$

then Rate of Change of angular momentum is Called Torque.

— Equilibrium of rigid body — A rigid body is one for which the distance between different particles do not change, even though they move under the influence of an external force.
A rigid body can execute two kind of motion.
(i) translatory motion (ii) Rotational motion.

— A rigid body is said to be in equilibrium of both the linear momentum and angular momentum of the rigid body remain constant with time. Hence the linear acceleration of its center of mass would be zero and the angular acceleration of the rigid body about any axis would be zero.

1 Translational equilibrium — The resultant of all the external forces acting on the body must be zero.
If the body at rest it is said to be in static equilibrium and if the body is in uniform motion along a straight path it is in dynamic equilibrium.

2 Rotational equilibrium — The resultant of torque due to all the forces acting on the body about any point must be zero.

a. Stable equilibrium — A body is said to be in stable equilibrium if tends to regain its equilibrium position after being slightly displaced and released.

b Unstable equilibrium — A body is said to be in unstable equilibrium if its gets further displaced from it equilibrium position after being slightly displaced and released.

c Neutral equilibrium — If a body stay in equilibrium position even after being slightly displaced and released, it is said to be neutral equilibrium.

— moment of inertia — the moment of inertia is define as a body rotating about a given axis tends to maintain its state of uniform rotation, unless an external torque is applied on it to change that state, this property of body by virtue of which it oppose the torque tending to change its state of rest or of uniform rotating about an axis is called rotational inertia or moment of Inertia.

The moment of inertia of a rigid body about a fixed axis is defined as the sum of the products of the masses of the particles constituting the body and the squares of their respective distance from the axis of rotation.

the S.I. units is $kg\,m^2$ and Diamension formula is $M^1L^2T^0$

The moment of inertia is depends on.

1. Mass of a body.
2. Size and shape of a body.
3. Distribution of mass about the axis of rotation.
4. Position and orientation of the axis of rotation w.r.t. the body.

— Relation between rotational kinetic energy and moment of inertia.

Consider a rigid body rotating about an axis with uniform angular velocity ω, the body may be assumed to consist of n particles of masses $m_1, m_2 \cdots m_n$ situated at distance $r_1, r_2 \cdots r_n$ from the axis of rotation. As the angular velocity ω of all the n particles is same so their linear velocities are.

$$v_1 = r_1\omega,\ v_2 = r_2\omega \ \cdots\cdots\ v_n = r_n\omega$$

Hence the totel K.E. of rotation of the body about axis is

$$= \frac{1}{2}m_1v_1^2 + \frac{1}{2}m_2v_2^2 + \cdots\cdots + \frac{1}{2}m_nv_n^2$$

$= \frac{1}{2} m_1 r_1^2 \omega^2 + \frac{1}{2} m_2 r_2^2 \omega^2 + \cdots\cdots + \frac{1}{2} m_n r_n^2 \omega^2.$

$= \frac{1}{2}(m_1 r_1^2 + m_2 r_2^2 + \cdots\cdots + m_n r_n^2)\omega^2 = \frac{1}{2}(\Sigma m r^2)\omega^2$

But $\Sigma m r^2 = I$, the moment of inertia of the body about the axis of rotation.

So Rotational K.E $= \frac{1}{2} I \omega^2$

— Radius of Gyration — It is defined as the distance from the axis of rotation at which, if the whole mass of the body were concentrated, its moment of inertia about the given axis would be same as which the actual distribution of mass.

or

The radius of gyration of a body about an axis of rotation be defined as the root mean square distance of its particles from the axis of rotation. It is denoted by K. and it depends on:

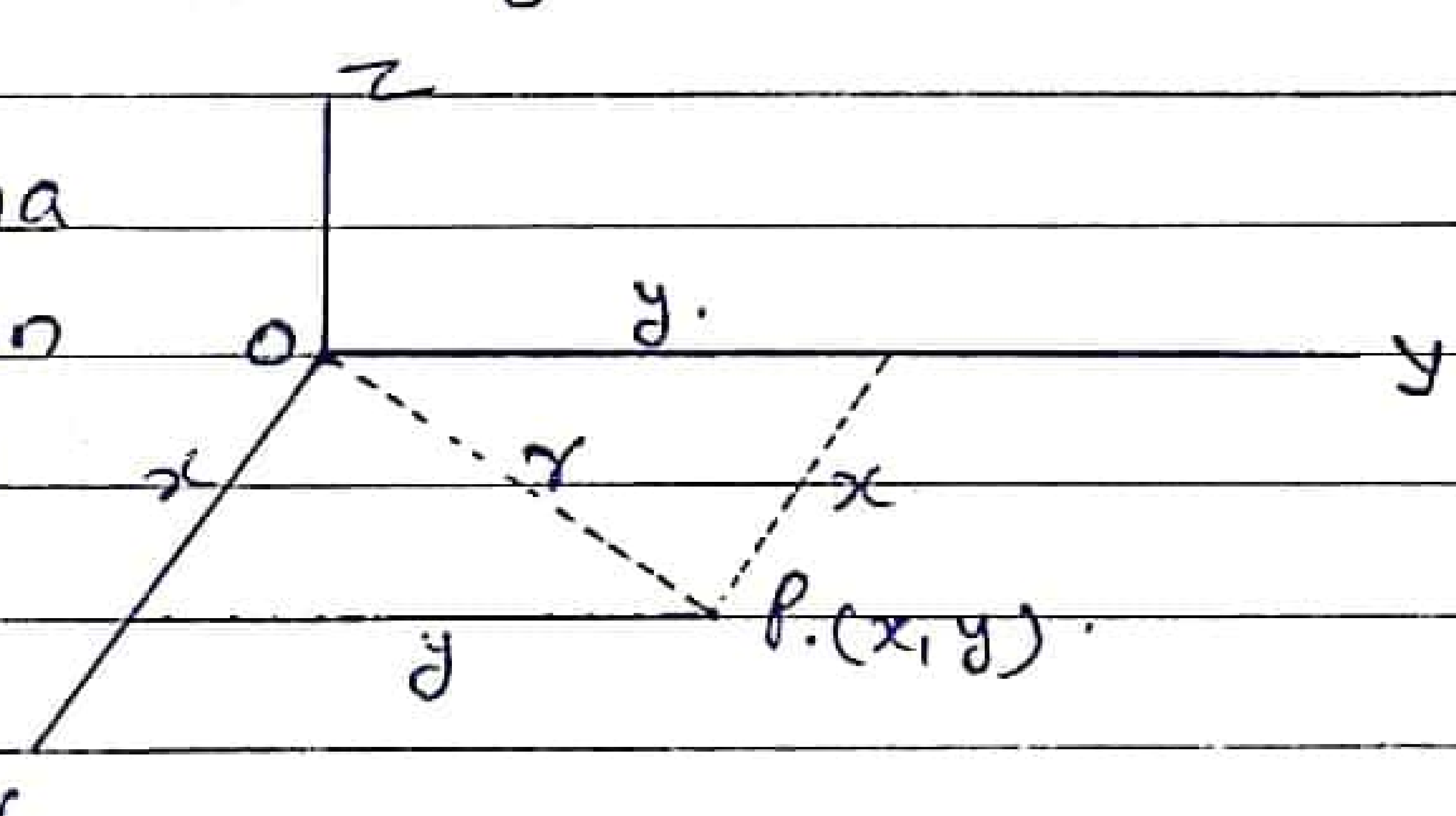
(I) Position and direction of the axis of rotation.

(II) Distribution of mass about the axis of rotation.

— Theorem of Perpendicular axes — The moment of inertia of a plane lamina about an axis perpendicular to its plane is equal to the sum of the moments of inertia of the lamina about any two mutually perpendicular axes in its own plane and intersecting each other at the point where the perpendicular axis passes through the lamina.

Proof —

Consider a plane lamina lying in the xOY plane. It can be assumed to be made up of large no. of particles. Let one such particle of mass m



at Point $P(x, y)$, clearly the distance of the Particle from x-, y-, z- axes are, y, x and r such that.

$$r^2 = x^2 + y^2.$$

Moment of inertia of the Particle about x axes.

$$= my^2.$$

Moment of inertia of whole lamina about x axes

$$I_x = \Sigma my^2.$$

and about y axes is $I_y = \Sigma mx^2.$

and about z axes is $I_z = \Sigma mr^2.$

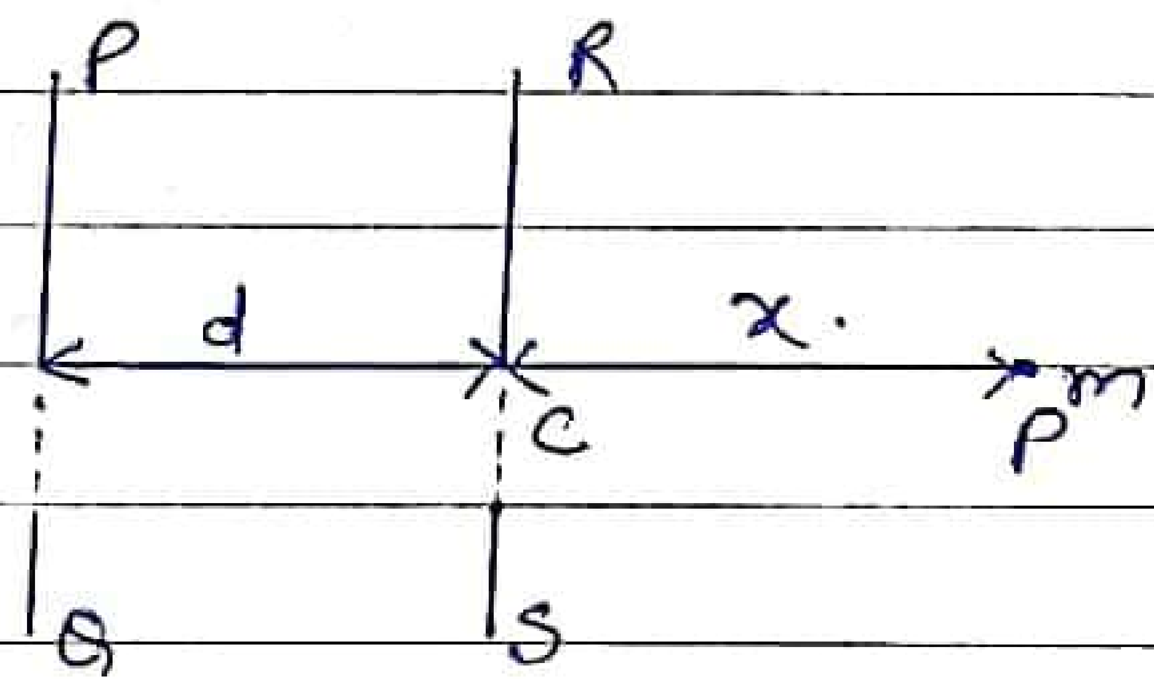
then $I_z = \Sigma mr^2 = \Sigma m(x^2+y^2)$

$$= \Sigma mx^2 + \Sigma my^2.$$

$$I_z = I_y + I_x.$$

— Theorem of Parallel axes – The moment of inertia of a body about any axis is equal to its moment of inertia about a Parallel axes through its Center of mass plus the Product of the mass of the body and the Square of the Perpendicular distance between the two Parallel axes.

Proof –



Let I be the moment of inertia of a body of mass M about an axis PQ. Let RS be the Parallel axis Passing through the Center of mass C of the body and at distance d from PQ. Let I_{cm} be the moment of Inertia of the body about the axis RS

Consider a Particle 'P' of mass 'm' at distance x from RS and so at distance $(x+d)$ from PQ.

Moment of Inertia of the Particle about axis PQ

$$= m(x+d)^2.$$

Moment of Inertia of the Whole body about axis PQ is.

$$= \Sigma m(x+d)^2 = \Sigma m(x^2+d^2+2xd)$$

$$= \Sigma mx^2 + \Sigma md^2 + \Sigma 2mxd.$$

Now $\Sigma mx^2 = I_{cm}$

$\Sigma md^2 = (\Sigma m)d^2 = Md^2$

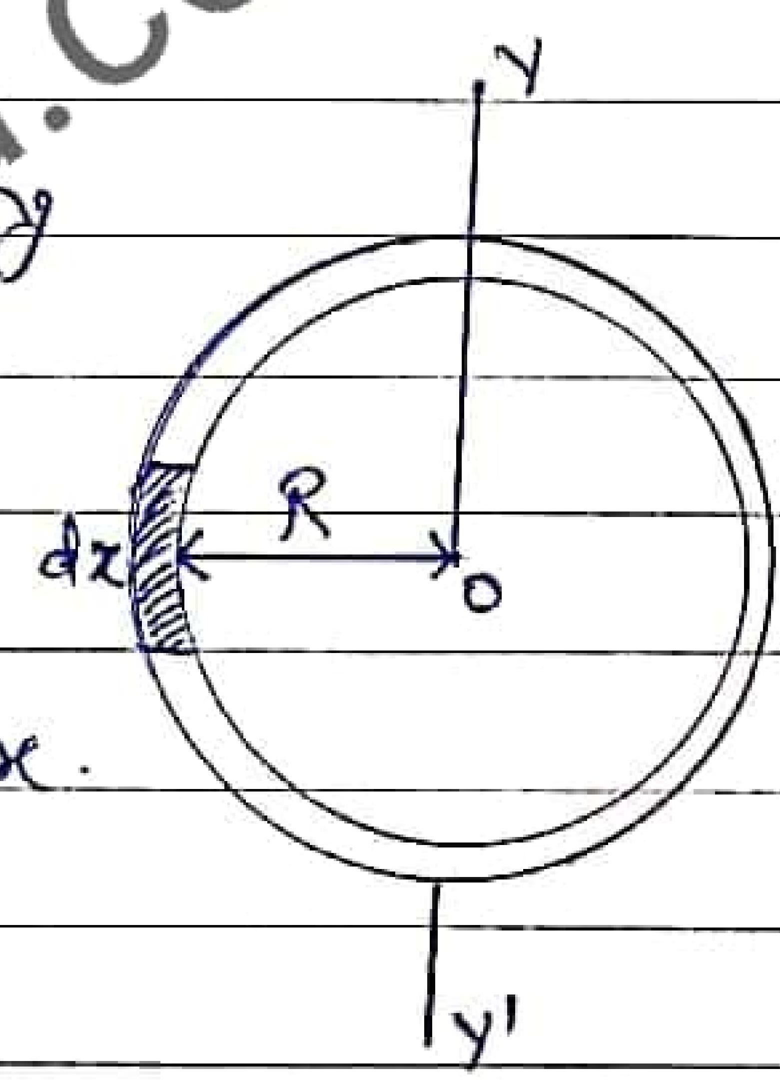
$\Sigma 2mxd = 2d\Sigma mx = 2d \times 0 = 0.$

because a body Can balance itself about its Center of mass so the sum of moments of masses of all its Particles about the axis RS is zero.

so $I = I_{cm} + Md^2$.

— Moment of Inertia of a thin Circular ring

a M.I. about an axis through its Center Perpendicular to its Plane.



Consider a thin uniform circular ring of radius R and mass m. The ring Can be imagined to be made of a large number of small elements.

Consider one such element of length dx.

Length of the Ring $= 2\pi R$.

mass per unit length of ring $= \frac{m}{2\pi R}$.

mass of the element dx $= \frac{m}{2\pi R} \times dx$.

moment of inertia of the small element about the axis YY'

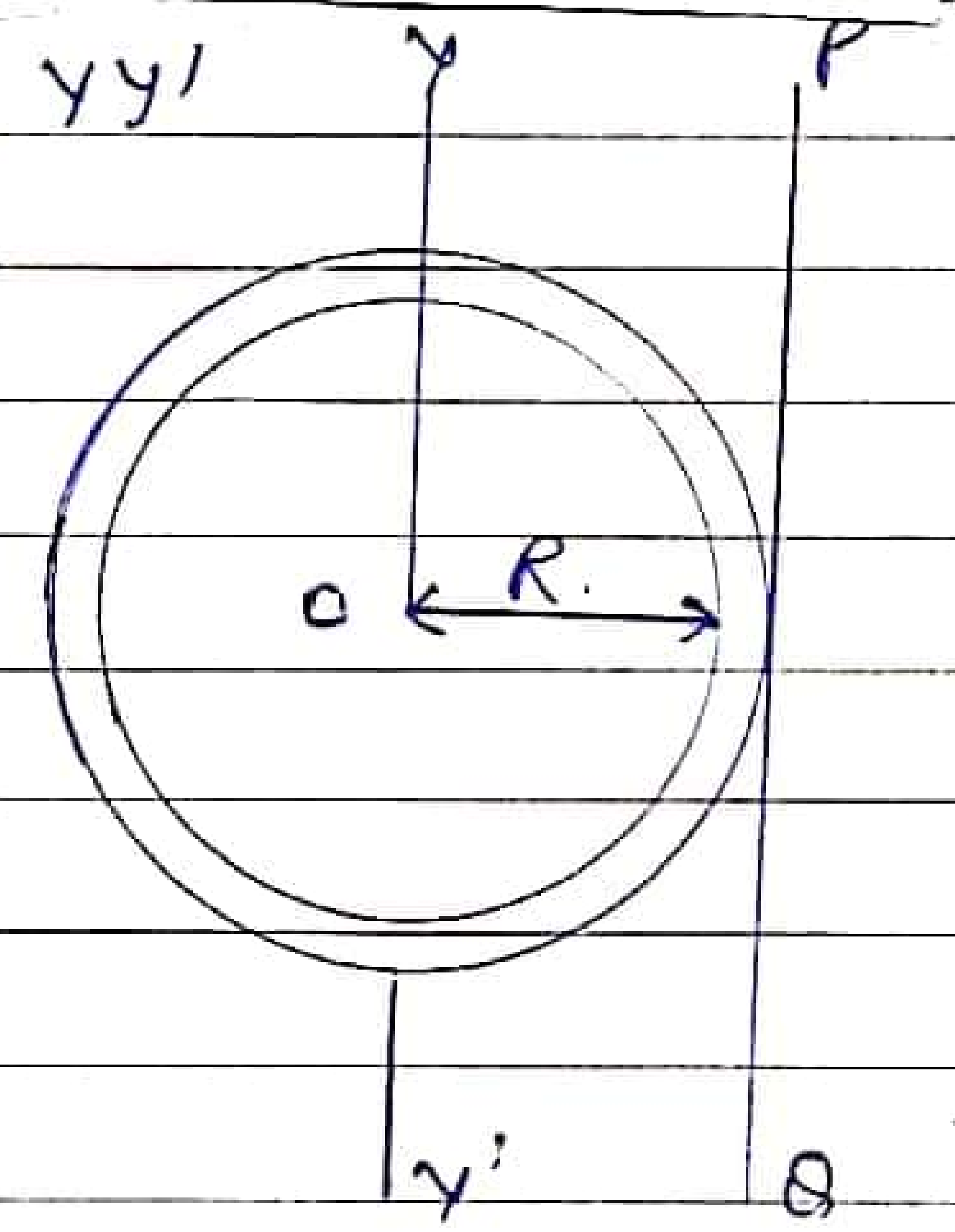
$$dI = \left(\frac{m}{2\pi R} . dx\right) \times R^2 = \frac{mR}{2\pi} . dx.$$

then moment of inertia of the whole Ring

$$I = \int_0^{2\pi R} \frac{mR}{2\pi} . dx = \frac{mR}{2\pi} \int_0^{2\pi R} dx = \frac{mR}{2\pi} (x)_0^{2\pi R}.$$

$$I = \frac{mR}{2\pi} \times 2\pi R = mR^2$$

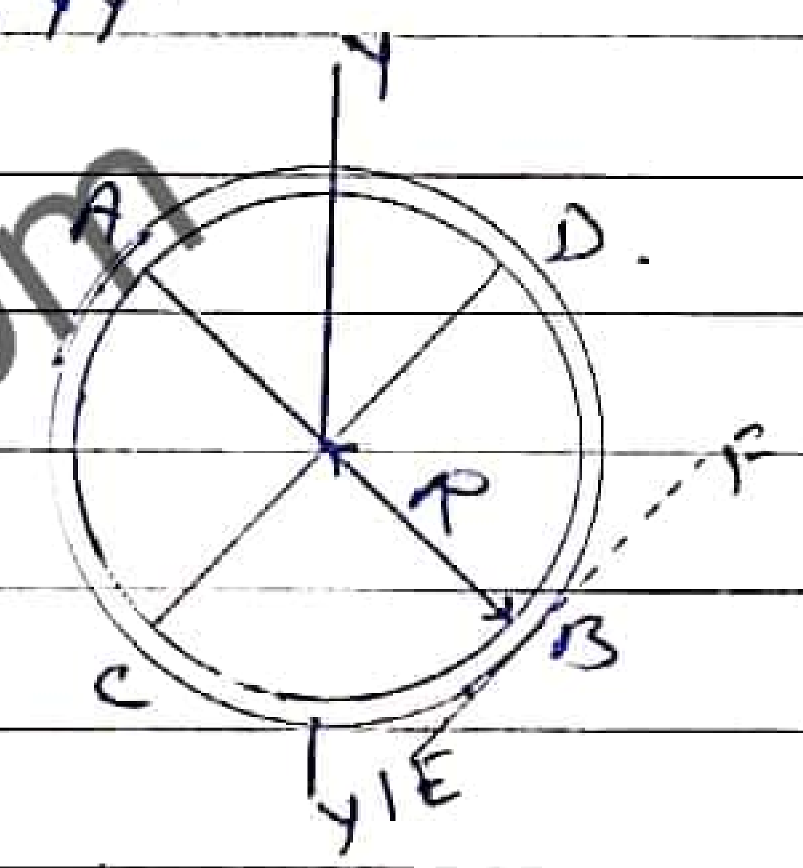
M.O.I of ring about a tangent Parallel to YY'



$$I_{PQ} = I_{YY'} + mR^2.$$
$$= mR^2 + mR^2.$$
$$= 2mR^2.$$

c. M.I. of ring about diameter Perpendiculer to YY'

According to Perpendiculer axis.



$$I_{AB} + I_{CD} = I_{YY'}$$
$$I_D + I_D = I_{YY'}$$
$$I_D = \frac{1}{2} I_{YY'} = \frac{1}{2} mR^2.$$

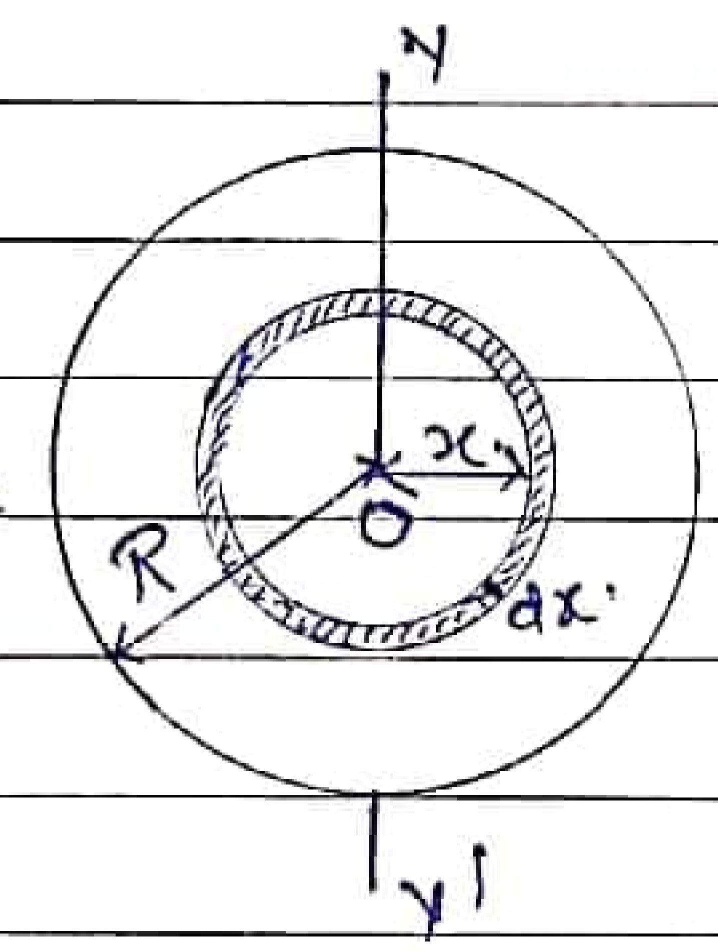
d. M.I. of ring about a tangent in its Plane.

According to Parallel axis.

$$I_{EF} = I_{CD} + mR^2.$$
$$= \frac{1}{2} mR^2 + mR^2 = \frac{3}{2} mR^2.$$

— moment of Inertia of a Uniform circular disc.

a. moment of inertia of a circular disc about an axis through its Center and PerPendiculer to its Plane —



Consider a Uniform disc of mass m and radius R, Let YY' is an axis Passing through the center O of the disc and Perpendiculer to the Plane.

Area of the disc $= \pi R^2$.

mass Per unit area of the disc $= \frac{m}{\pi R^2}$.

We Can Imagine the disc to be made up of a large no of Concentric rings. Let we Consider a ring of radius x and.

width dx.

Area of the ring = Circumference x width.

$= 2\pi x \times dx$.

mass of the Concentric ring $M = 2\pi x \cdot dx \times \frac{m}{\pi R^2} = \frac{2mx\,dx}{R^2}$.

moment of Inertia of Ring

$$dI = Mx^2 = \frac{2mx\,dx}{R^2} \cdot x^2.$$

$$= \frac{2mx^3}{R^2} \cdot dx.$$

Total moment of inertia of disc $= \int_0^R \frac{2mx^3}{R^2} \cdot dx$.

$$= \frac{2m}{R^2}\left(\frac{x^4}{4}\right)_0^R = \frac{2m}{R^2} \cdot \frac{R^4}{4} = \frac{1}{2}mR^2.$$

b. moment of inertia about a tangent Parallel to yy'

$$= \frac{1}{2}mR^2 + mR^2 = \frac{3}{2}mR^2.$$

(c) moment of Inertia about diameter Perpendicular to yy'

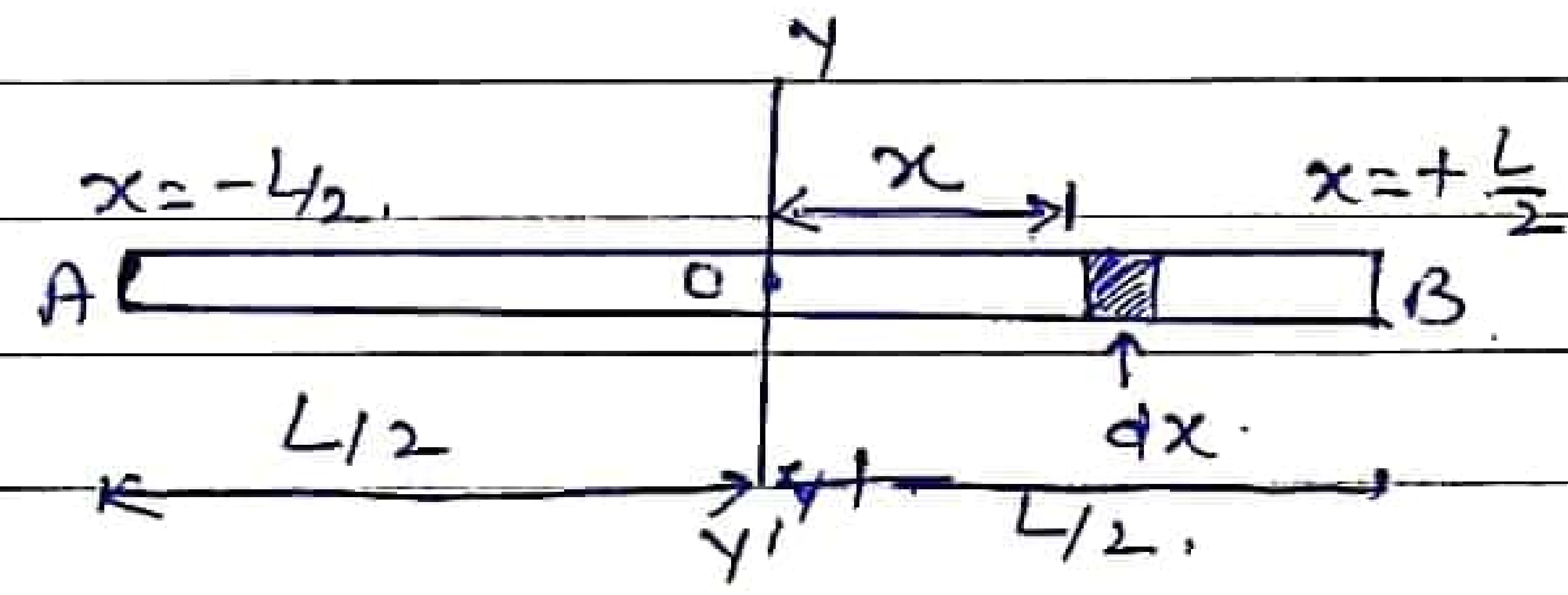
$I_D + I_D = \frac{1}{2}mR^2$ or $I_D = \frac{1}{4}mR^2$.

(d) moment of inertia about a tangent Parallel to diameter.

$$= \frac{1}{4}mR^2 + mR^2 = \frac{5}{4}mR^2.$$

— Moment of inertia of a thin uniform Rod.

(a) M.I. of a thin uniform rod about a Perpendicular axis through its center.



Consider a thin uniform rod AB of length L and mass m, free to rotate about an axis YY' through its center O and Perpendicular to its length.

mass Per unit length of the Rod $= \frac{m}{L}$

Consider a small mass element of length dx at a distance x from O

mass of the small element $= \frac{m}{L}.dx$.

moment of inertia of small element about YY'

$$dI = \frac{m}{L}.dx.x^2.$$

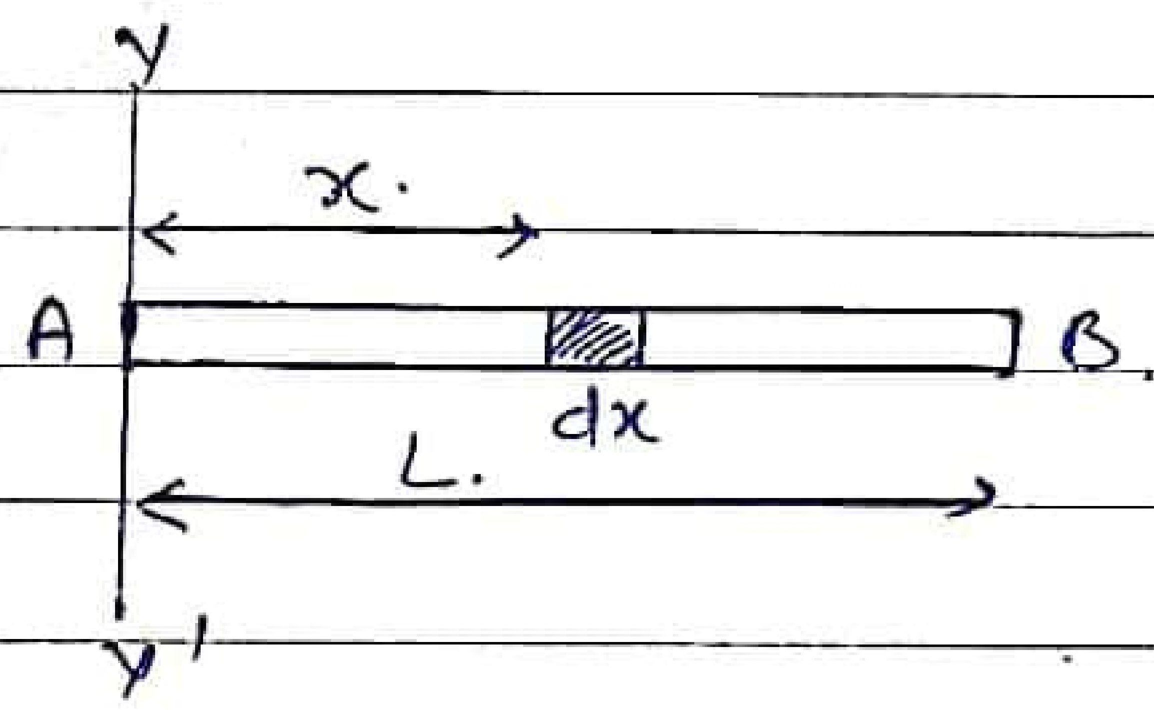
moment of inertia of the whole rod about YY'

$$I = \int dI = \int_{-L/2}^{L/2} \frac{m}{L}.x^2.dx = \frac{m}{L}.\left(\frac{x^3}{3}\right)_{-L/2}^{L/2}$$

$$= \frac{m}{3L}\left[\frac{L^3}{8} - \left(-\frac{L^3}{8}\right)\right] = \frac{m}{L}\frac{L^3}{12} = \frac{mL^2}{12}$$

b moment of inertia of a thin uniform rod about a perpendicular axis through its one end.

Consider a thin uniform rod AB of length L and mass m free to rotate about an axis passing through its on end A. YY' perpendicular to its length.



mass per unit length $= \frac{m}{L}$.

Consider a small element of length dx of the rod at a distance x from A.

mass of small element $= \frac{m}{L}.dx$.

moment of inertia $dI = \frac{m}{L}.dx.x^2$.

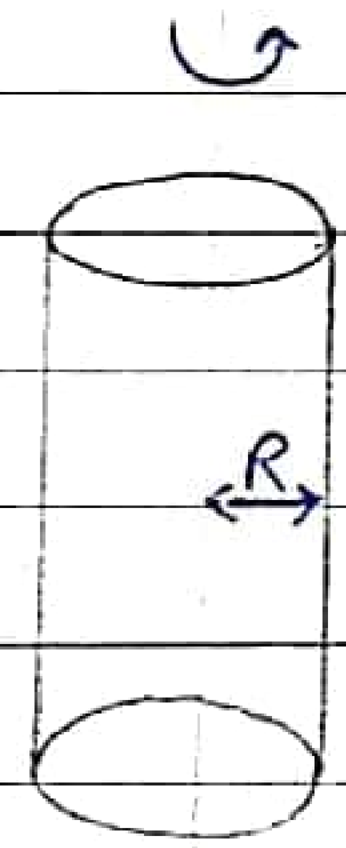
Total moment of inertia $I = \int dI = \int_0^L \frac{m}{L}.x^2.dx$.

$$= \frac{m}{L}\left(\frac{x^3}{3}\right)_0^L = \frac{m}{L}.\frac{L^3}{3} = \frac{mL^2}{3}$$

Moment of inertia of a cylinder

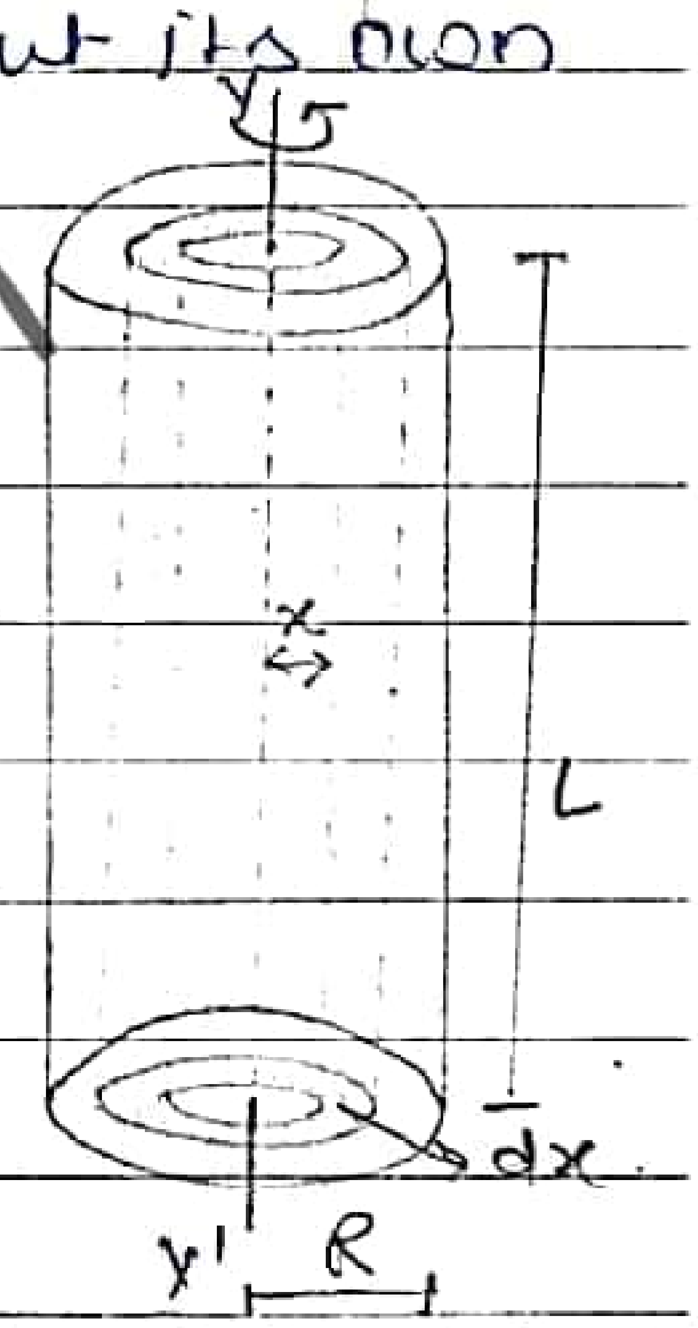
(a) M.I. of a hollow cylinder about its own axis.

Consider a hollow cylinder of mass m and radius R every element of the cylinder is at the same perpendicular distance R from its axis.



So $I = \int R^2\, dm = R^2 \int dm = mR^2$.

(b) moment of inertia of uniform solid cylinder about its own axis.



Consider a solid cylinder of mass m and radius R and length L, moment of inertia about its own axis YY'

Volume of cylinder $= \pi R^2 L$

mass per unit volume of cylinder

$$\rho = \frac{m}{\pi R^2 L}$$

we can imagine the cylinder to be made of a large no. of coaxial cylinderical shell. Consider one such shell of Inner radius x and outer radius $x+dx$. The cross section of the shell is a ring of radius x and thickness dx.

Cross section area of shell = Circumference × thickness

$= 2\pi x . dx$.

Volume of the shell = cross section area × length.

$= 2\pi x . dx \times L$.

mass of shell $M = 2\pi x . dx \times L \times \rho$

$$= 2\pi x . dx . \times L \times \frac{m}{\pi R^2 L} = \frac{2m}{R^2} . x . dx .$$

moment of inertia of shell

$$dI = M . x^2 = \frac{2m}{R^2} x . dx \times x^2 = \frac{2m}{R^2} . x^3 . dx .$$

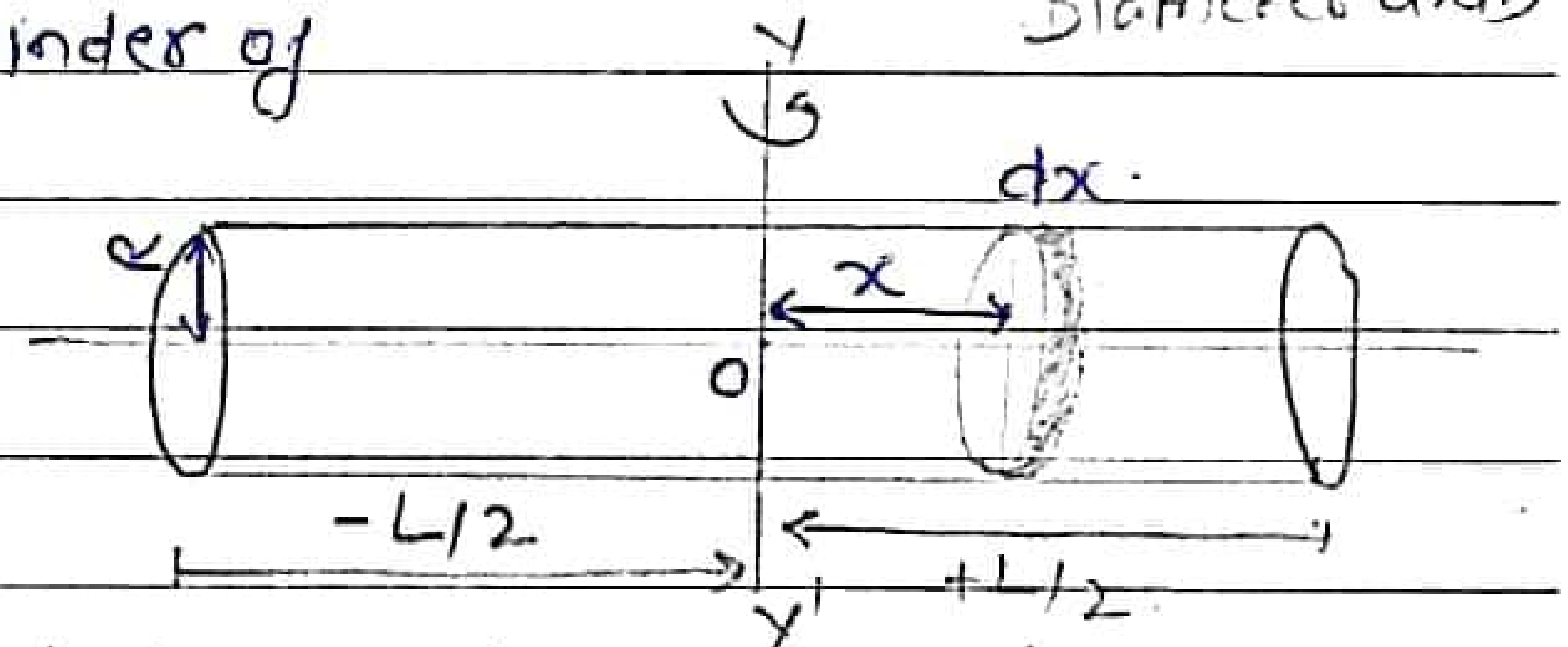
then moment of Inertia of total cylinder $I = \int_0^R \frac{2m}{R^2} . x^3 . dx$.

$$\frac{2m}{R^2}\left[\frac{x^4}{4}\right]_0^R = \frac{2m}{R^2}\cdot\frac{R^4}{4} = \frac{1}{2}mR^2.$$

c. moment of inertia of a solid cylinder about an axis through its center and perpendicular to its axis.

Consider a uniform solid cylinder of mass m, radius R and length L,

mass per unit length $= \frac{m}{L}$



we can imagine the cylinder to be made up of a large no. of thin circular disc placed perpendicular to the axis of cylinder.

Consider one such disc of thickness dx and at a distance x from the center. O

$$\text{mass of disc} = \frac{m}{L}.dx$$

moment of inertia of disc about diameter

$$= \frac{1}{4}\frac{m}{L}dx.R^2 = \frac{mR^2}{4L}.dx$$

According to the parallel axis inertia about YY'

$$dI = \frac{mR^2}{4L}.dx + \frac{m}{L}.dx.x^2.$$

$$= \frac{m}{L}\left(\frac{R^2}{4}+x^2\right).dx.$$

moment of inertia of the cylinder

$$I = \int_{-L/2}^{L/2} \frac{m}{L}\left(\frac{R^2}{4}+x^2\right).dx.$$

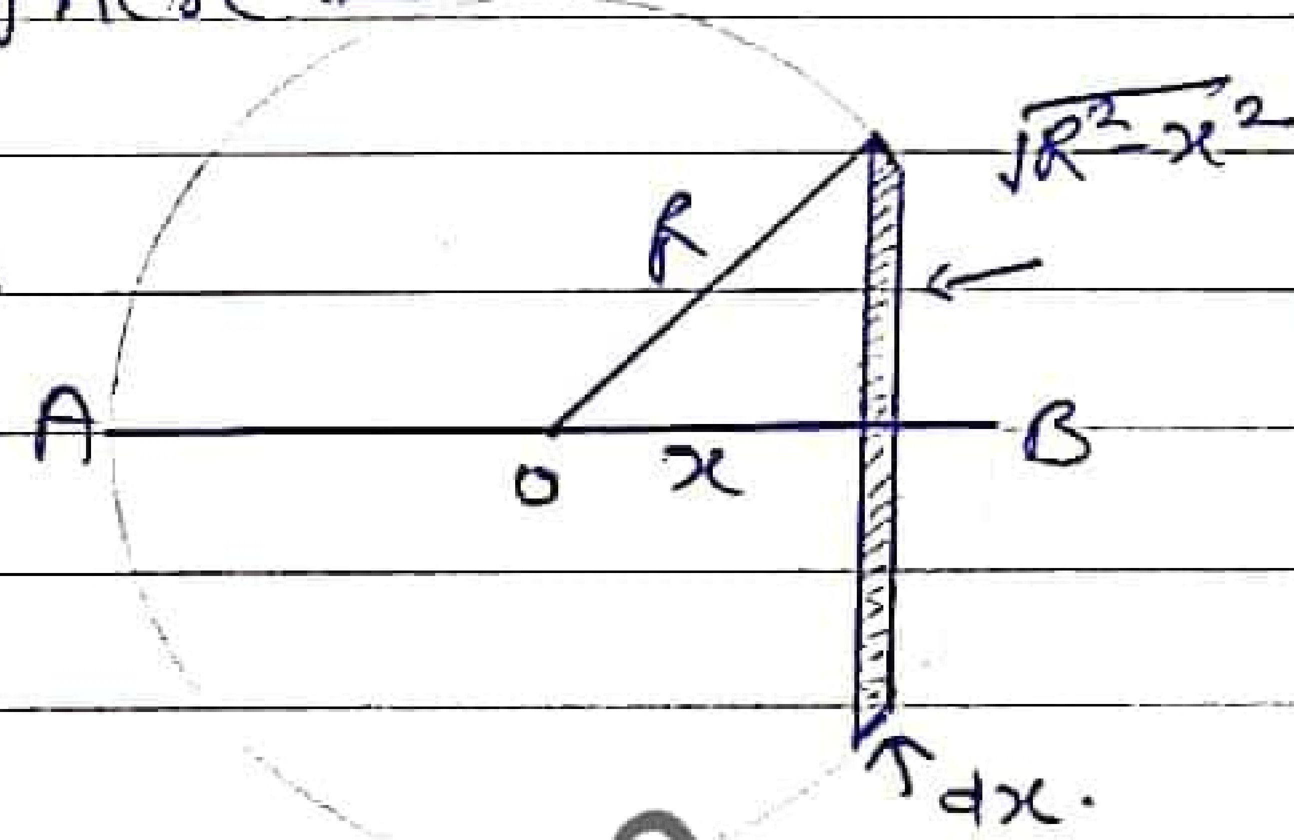
$$= 2\int_0^{L/2} \frac{m}{L}\frac{R^2}{4}.dx + 2\int_0^{L/2}\frac{m}{L}.x^2.dx.$$

$$= \frac{2mR^2}{4L}.[x]_0^{L/2} + \frac{2m}{L}.\left(\frac{x^3}{3}\right)_0^{L/2}.$$

$$= \frac{2m}{L}\left[\frac{R^2}{4}\frac{L}{2} + \frac{L^3}{24}\right]$$

$$= m\left[\frac{R^2}{4} + \frac{L^2}{12}\right]$$

— Moment of inertia of a solid sphere —



Q moment of inertia of a solid sphere about its diameter —

Consider a uniform solid sphere of mass m and Radius R

Volume of sphere $= \frac{4}{3}\pi R^3$.

mass per unit volume $\rho = \frac{m}{\frac{4}{3}\pi R^3} = \frac{3m}{4\pi R^3}$

we can imagine the sphere to be made up of a large number of thin slices placed perpendicular to the diameter AB. Consider one such slice of thickness dx placed at distance x from center O

then radius of slice $= \sqrt{R^2-x^2}$.

Volume of slice $=$ Area x thickness

$= \pi(R^2-x^2) \times dx$.

mass of slice $= \pi(R^2-x^2)\,dx \times \frac{3m}{4\pi R^3} = \frac{3m(R^2-x^2)dx}{4R^3}$.

moment of inertia of slice. $dI = \frac{1}{2} \times \text{mass} \times (\text{radius})^2$.

$$dI = \frac{1}{2} \times \frac{3m(R^2-x^2)}{4R^3}.dx \cdots (\sqrt{R^2-x^2})^2.$$

$$= \frac{3m}{8R^3}(R^2-x^2)^2.dx.$$

then moment of inertia of total sphere.

$$I = 2\int dI = 2\int_0^R \frac{3m}{8R^3}(R^2-x^2)^2 \cdot dx.$$

$$\frac{6m}{8R^3}\int_0 (R^4+x^4-2R^2x^2)\,dx.$$

$$=\frac{3m}{4R^3}\left[R^4x+\frac{x^5}{5}-2.R^2.\frac{x^3}{3}\right]_0^R$$

$$=\frac{3m}{4R^3}\left[R^5+\frac{R^5}{5}-\frac{2R^5}{3}\right]=\frac{3m}{4R^3}\left[\frac{15R^5+3R^5-10R^5}{15}\right]$$

$$=\frac{\cancel{3}m}{\cancel{4}R^3}\cdot\frac{\cancel{8}^{2}R^5}{\cancel{15}5}=\frac{2}{5}mR^2$$

b moment of inertia about tangent Perpendicular to diameter:

$$I_T=\frac{2}{5}mR^2+mR^2=\frac{7}{5}mR^2.$$

— Relation between torque and moment of inertia —

When a torque acts on a body capable of rotation about an axis, it Produce an angular acceleration in the body. If the angular velocity of each Particle is ω then the angular acceleration $\alpha=\frac{d\omega}{dt}$ will be same for all Particles of the body.

The linear acceleration will depend on their distance $r_1, r_2, \dots r_n$ from the axis of rotation.

Let a Particle P of mass m_1 at a distance r_1 from the axis of rotation, let its linear velocity be v_1

Linear acceleration $a_1=r_1\alpha$

Force acting on Particle $F_1=m_1r_1\alpha$.

moment of force F_1 about the axis of rotation is

$$\tau_1=F_1\times r_1=m_1r_1^2\alpha$$

Total torque acting on the rigid body is.

$$\tau=\tau_1+\tau_2+\dots+\tau_n$$

$$=m_1r_1^2\alpha+m_2r_2^2\alpha+\dots+m_nr_n^2\alpha.$$

$$=(m_1r_1^2+m_2r_2^2+\dots+m_nr_n^2)\alpha.$$

$$=(\Sigma mr^2)\alpha$$

But $\Sigma mr^2=I$

than $\tau = I\alpha$.

— Relation between Angular momentum and moment of inertia and angular velocity —

Consider a rigid body rotating about a fixed axis with uniform angular velocity ω. The body consist of n particles of masses $m_1, m_2 \cdots m_n$, situated at distance $r_1, r_2 \cdots r_n$ from the axis of rotation. The angular velocity of n particles will be same but their linear velocities will be different.

$V_1 = r_1\omega$, $V_2 = r_2\omega$ $\cdots\cdots$ $V_n = r_n\omega$

Linear momentum $P_1 = m_1V_1 = m_1r_1\omega$.

$P_n = m_nV_n = m_nr_n\omega$

moment of linear moment of particles about the fixed axis.

$L_1 = P_1r_1 = m_1r_1^2\omega$ $\cdots\cdots$ $L_n = P_nr_n = m_nr_n^2\omega$

then total moment of linear moment of n particles.

$L = L_1 + L_2 + \cdots\cdots + L_n$.

$= m_1r_1^2\omega + m_2r_2^2\omega + \cdots\cdots + m_nr_n^2\omega$.

$= (m_1r_1^2 + m_2r_2^2 + \cdots\cdots + m_nr_n^2) \times \omega$. $= (\Sigma mr^2) \times \omega$

$L = I\omega$

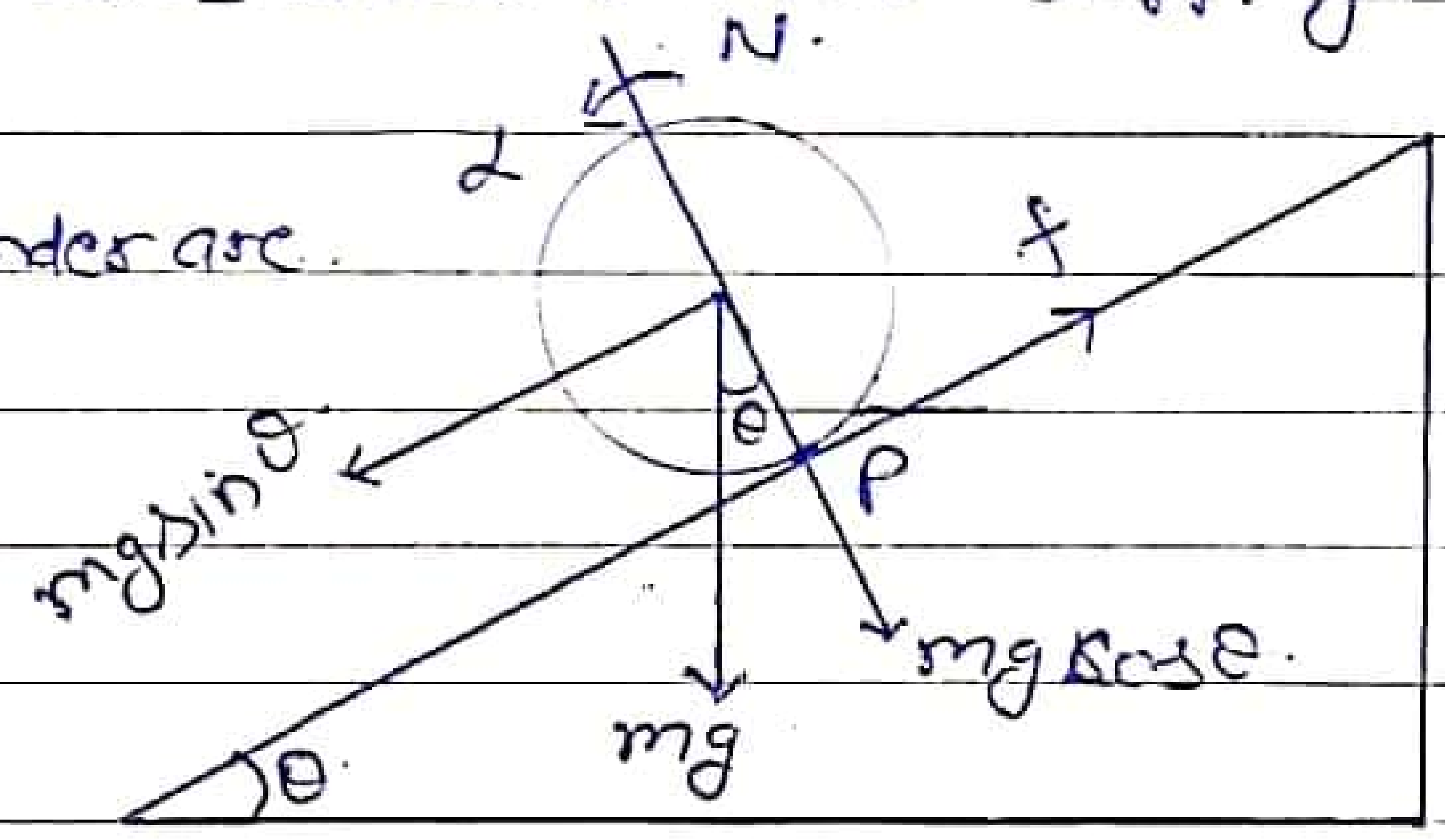
— Law of Conservation of angular momentum — when the total external torque acting on a rigid body is zero, the total angular momentum of the body is conserved. this is the law of Conservation of angular momentum.

when $\tau = 0$ then $\frac{dL}{dT} = 0$ so L is Constant.

so $I_1\omega_1 = I_2\omega_2$.

when no external torque acting, the angular velocity ω of the body can increased or decreased by decreasing or increasing the moment of inertia of the body.

— Solid cylinder rolling down an inclined plane without Slipping — Consider a solid cylinder of mass m and radius R rolling down a plane inclined at an angle θ to the horizontal. Suppose the cylinder rolls down without slipping.



the external force acting on cylinder are.

(I) the weight mg, acting vertically down ward.

(II) the normal Reaction N acting Perpendiculer to the Plane

(III) The friction force f acting upward and Parallel to the inclined Plane.

$$N = mg\cos\theta. \qquad \text{—(I)}$$

$$F = mg\sin\theta - f. \qquad \text{—(II)}$$

it is only the force of friction f which exerts torque τ on the cylinder and makes it rotate with angular acceleration α. It acts tangentially at the Point of contact P and has lever arm equal to radius R.

$$\tau = \text{Force} \times \text{Arm} = f \times r$$

$$I\alpha = fR$$

$$f = \frac{I\alpha}{R} = \frac{Ia}{R^2}. \qquad \because \alpha = \frac{a}{R}.$$

Putting the value of f in eq. (II).

$$ma = mg\sin\theta - \frac{Ia}{R^2}.$$

$$a = g\sin\theta - \frac{Ia}{mR^2}.$$

$$a + \frac{Ia}{mR^2} = g\sin\theta.$$

$$a = \frac{g\sin\theta}{1 + \frac{I}{mR^2}} \qquad \because I = \tfrac{1}{2}mR^2.$$

$$a = \frac{g\sin\theta}{1 + \frac{\frac{1}{2}mR^2}{mR^2}} = \frac{2g\sin\theta}{3}$$

For hollow cylinder $I = mR^2$ so $a = \frac{1}{2}g\sin\theta$.

Putting the value of a in eq. (ii).

$$f = mg\sin\theta - m \cdot \frac{2}{3} g\sin\theta = \frac{1}{3} mg\sin\theta.$$

If μ_s is the coefficient of friction between the cylinder and the inclined plane then.

$$\mu_s = \frac{f}{N} = \frac{\frac{1}{3} mg\sin\theta}{mg\cos\theta} = \frac{1}{3}\tan\theta.$$

To prevent slipping, the coefficient of static friction must be equal to or greater than.

$$\mu_s \geq \frac{1}{3}\tan\theta.$$

Q. A ring, a disc and a sphere all of the same radius and mass roll down an inclined plane from the same height h which of three reaches the bottom earliest and latest.

An. The acceleration of a body rolling down an inclined plane is.

$$a = \frac{mg\sin\theta}{m + I/R^2} = g\sin\theta\left(\frac{mR^2}{mR^2 + I}\right)$$

For a Ring $I = mR^2$.

$$\text{so } a_{Ring} = g\sin\theta\left(\frac{mR^2}{mR^2 + mR^2}\right) = \frac{1}{2} g\sin\theta.$$
$$= .5 g\sin\theta.$$

For a disc $I = \frac{1}{2} mR^2$.

$$\text{so } a_{disc} = g\sin\theta\left(\frac{mR^2}{mR^2 + \frac{1}{2}mR^2}\right) = \frac{2}{3} g\sin\theta$$
$$= .67 g\sin\theta.$$

For a sphere $I = \frac{2}{5} mR^2$.

$$\text{so } a_{sp} = g\sin\theta\left(\frac{mR^2}{mR^2 + \frac{2}{5}mR^2}\right) = \frac{5}{7} g\sin\theta$$
$$= .71 g\sin\theta$$

So the sphere reaches the bottom earliest and Ring latest.

— Motion of a mass point attached to a string wounded on a cylinder.— Consider a solid cylinder of mass m and radius R. It is mounted on a frictionless horizontal axle so that it can freely rotate about it axis. A light string is wound round the cylinder and mass M is suspended from it when the mass M is released from rest, it move down with acceleration a. Let T be the tension in the string.

(a) Linear acceleration a of the Point mass –

According to newton's second law, the net downward force on the Point mass is.

$$Ma = Mg - T. \quad -(I)$$

the tension T of the string acts tangentially on the cylinder and Produce a torque τ

$$\tau = T.R \quad -(II)$$

If I is the moment of inertia of the cylinder and α is the angular acceleration Produce in it then.

$$\tau = I\alpha. \quad -(III).$$

By (II) and (III).

$$I\alpha = TR \text{ then } T = \frac{I\alpha}{R} = \frac{Ia}{R^2} \quad -④ \left[\because \alpha = \frac{a}{R}\right]$$

Putting the value of T in eq. (I).

$$Ma = Mg - \frac{Ia}{R^2}.$$

$$Ma\left(1 + \frac{I}{MR^2}\right) = Mg.$$

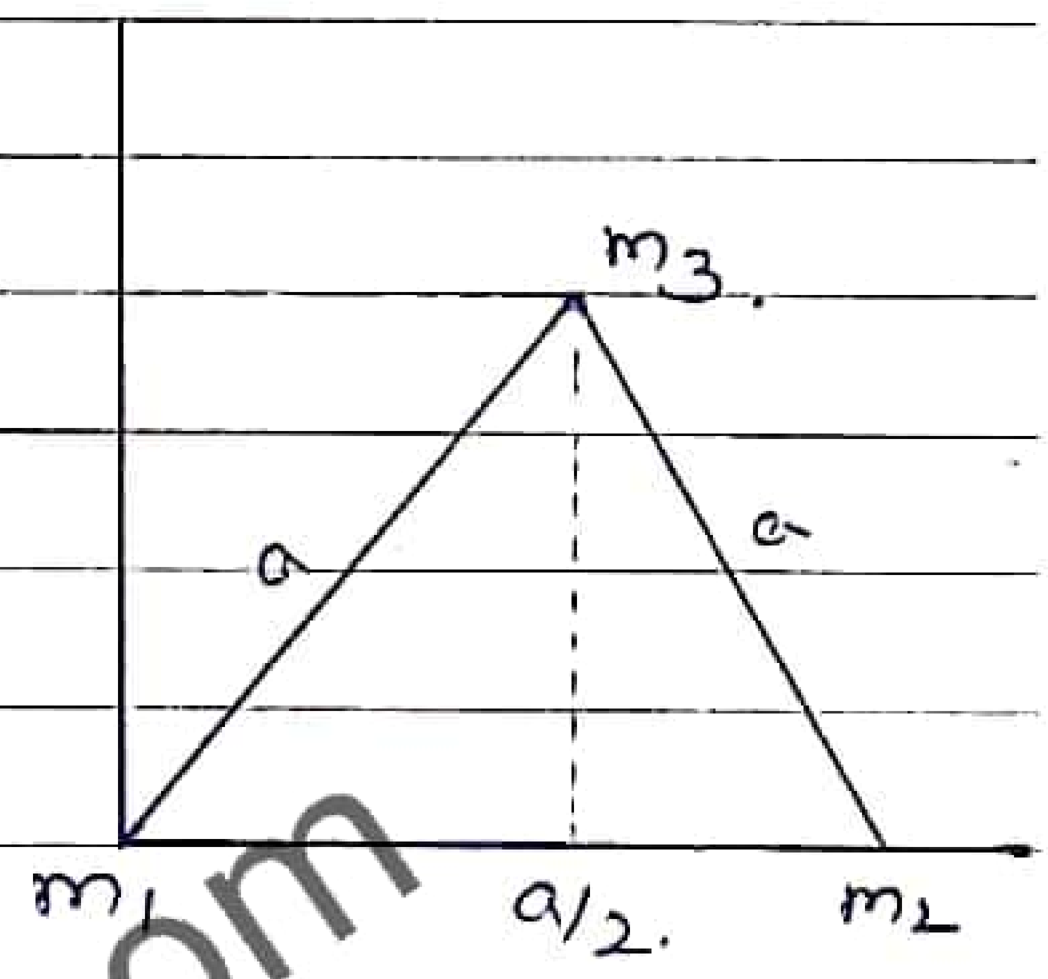
$$a = \frac{g . MR^2}{(MR^2 + I)} \quad — ⑤.$$

$$\alpha = \frac{a}{R} = \frac{g . MR^2}{MR^2 + I} \times \frac{1}{R} = \frac{gMR}{(MR^2 + I)}$$

$$T = \frac{Ia}{R^2} = \frac{I}{R^2} . \frac{g . MR^2}{(MR^2 + I)} = \frac{IgM}{(MR^2 + I)}$$

Q. If three point masses m_1, m_2 and m_3 are situated at the vertices of an equilateral triangle of side a, then what will be the co-ordinates of the center of mass of this system.

Ans: Let the point m_1 lie at the origin of the co-ordinate system then.



Co-ordinate of $m_1 = (0, 0)$

Co-ordinate of $m_2 = (a, 0)$.

Co-ordinate of $m_3 = \left(\frac{a}{2}, \frac{\sqrt{3}}{2}a\right)$

$$\therefore x_{cm} = \frac{m_1x_1 + m_2x_2 + m_3x_3}{m_1 + m_2 + m_3}$$

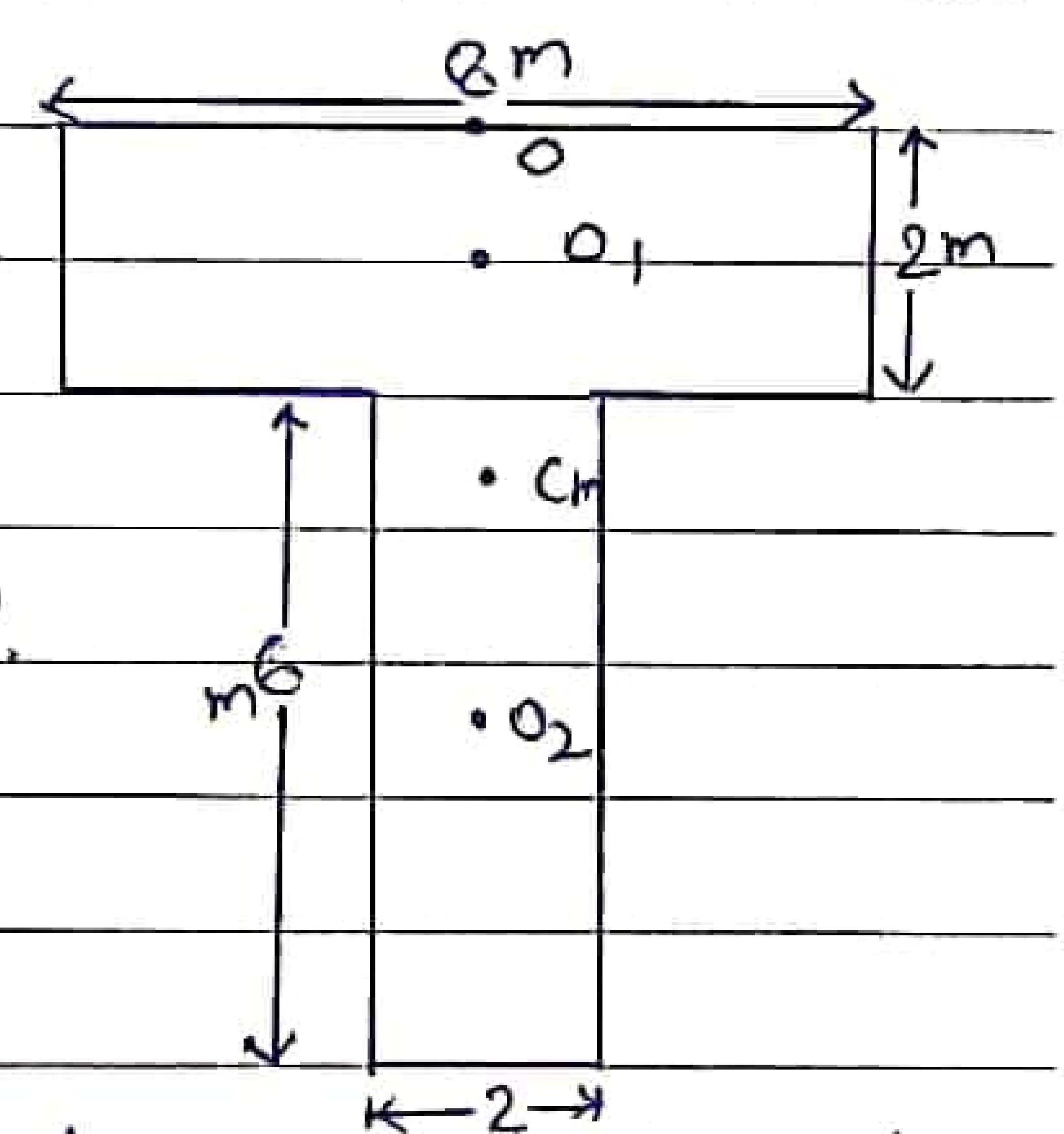
$$= \frac{m_1 \times 0 + m_2 \times a + m_3 \times a/2}{m_1 + m_2 + m_3}$$

$$= \frac{a m_2 + \frac{a}{2} m_3}{m_1 + m_2 + m_3}$$

$$Y_{cm} = \frac{m_1y_1 + m_2y_2 + m_3y_3}{m_1 + m_2 + m_3} = \frac{m_1 \times 0 + m_2 \times 0 + m_3 \times \frac{\sqrt{3}}{2}a}{m_1 + m_2 + m_3}$$

$$= \frac{\frac{\sqrt{3}}{2} a m_3}{m_1 + m_2 + m_3}$$

Q. Find the position of center of mass of the figure



Ans. Let mass per unit area of the plate be ρ.

Mass of horizontal portion $= 8 \times 2 \times \rho$
$= 16\rho$.

Mass of vertical portion $= 6 \times 2 \times \rho = 12\rho$.

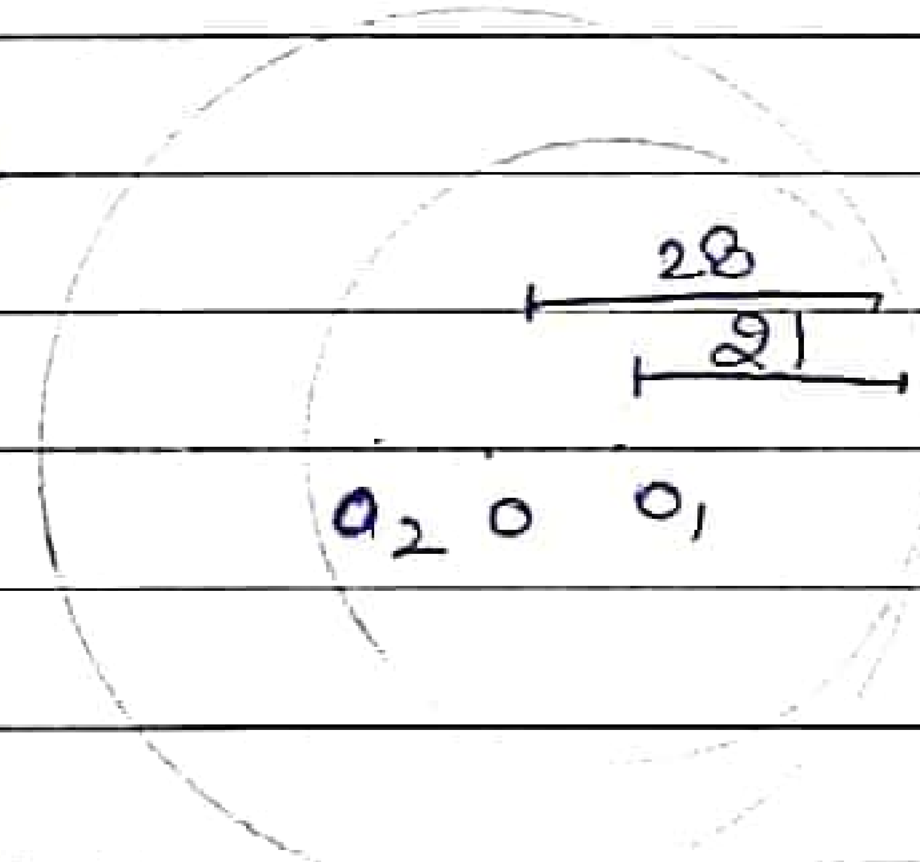
Center of mass O_1 is at a distance 1 m from the top and center of mass O_2 is at a distance $2+3 = 5$ m from the top.

Center of mass C_m from the top

$$= \frac{1 \times 16\rho + 5 \times 12\rho}{16\rho + 12\rho} = \frac{16\rho + 60\rho}{28\rho} = \frac{76\rho}{28\rho} = 2.71\text{ m}$$ from top.

Circular plate of uniform thickness has a diameter of 56 cm. A circular portion of diameter 42 cm is removed from one edge of the plate. Find C.M. of the remaining portion.

Ans. Let m be the mass per unit area of the plate.



Mass of original plate $= m_1 = \pi (28)^2 \times m$.

mass of circular portion removed $m_2 = \pi (21)^2 \times m$.

mass of shaded portion.

$$M = m_1 - m_2 = \pi \times m [(28)^2 - (21)^2]$$
$$= \pi \times m \times (28+21)(28-21)$$
$$M = \pi \times m \times 49 \times 7 = 343 \pi m.$$

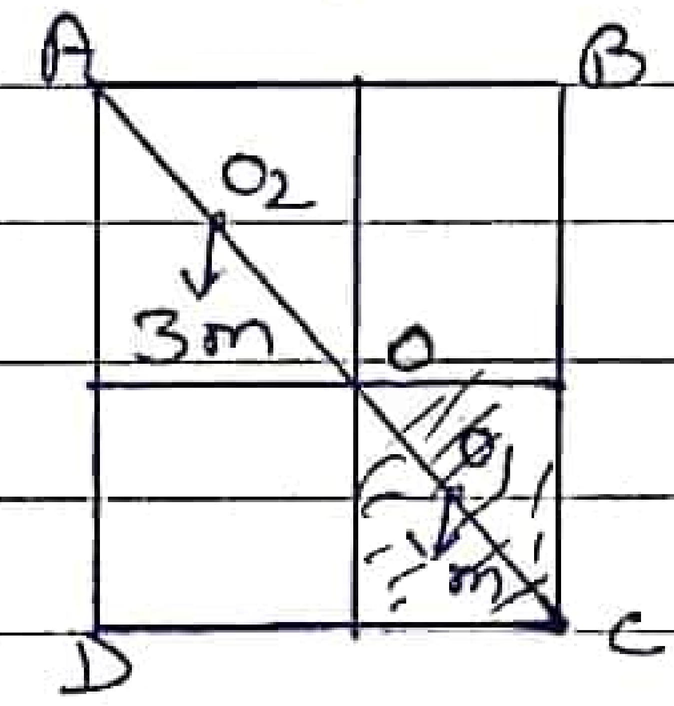
masses m_2 and M may be assumed to be concentrated at O_1 and O_2 respectively and O is their CM.

momentum about O_1 of m_2 = momentum of M about O_2

$$m_2 \times OO_1 = M \times OO_2.$$
$$OO_2 = \frac{m_2}{M} \times OO_1 = \frac{441 \pi m}{343 \pi m} \times (28-21)$$
$$= 9 \text{ cm}.$$

Q. A square of side a cm and uniform thickness is divided into four equal squares, If one of the square is cut off, find the position of the center of mass of the remaining portion from O

Ans. Let mass of each small square be m. then total mass of the square will be $4m$ which acts at its center of mass O. Let O_1 be CM of cut off square and O_2 be CM of the remaining portion of mass $3m$.



then $AC = \sqrt{AB^2 + BC^2} = \sqrt{a^2+a^2} = a\sqrt{2}$.

$$OC = \frac{1}{2} AC = \frac{1}{2} a\sqrt{2} = a/\sqrt{2}.$$
$$OO_1 = \frac{1}{2} OC = \frac{1}{2} \cdot \frac{a}{\sqrt{2}} = \frac{a}{2\sqrt{2}}.$$

Momentum of unshaded Portion about O

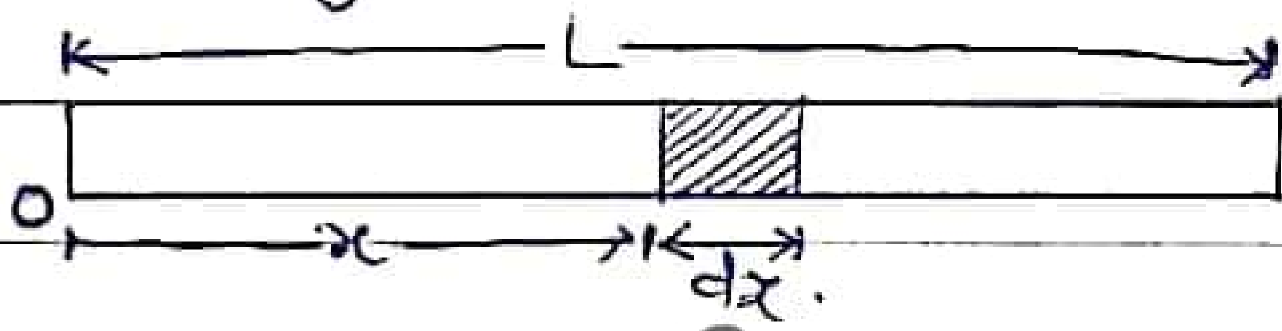
= momentum of shaded Portion about O.

$3m \times OO_2 = m \times OO_1$

$OO_2 = \frac{m}{3m} \times OO_1 = \frac{1}{3} \times \frac{a}{2\sqrt{2}} = \frac{a}{6\sqrt{2}}$.

Q. Show that the Center of mass of a uniform rod of mass M and length L lies at the middle Point of the rod.

Ans. Suppose the rod is Placed along x-Axis with its left end at the origin O. Consider a small element of thickness dx at distance x from O



mass of small element $dm = \frac{M}{L} dx$

Position of center of mass

$$x_{cm} = \frac{1}{M}\int_0^L x . dm = \frac{1}{M}\int \frac{M}{L} . x . dx .$$

$$= \frac{1}{L}\int_0^L x\, dx = \frac{1}{L}\left[\frac{x^2}{2}\right]_0^L = \frac{1}{L} . \frac{L^2}{2} = \frac{L}{2}$$

Q. Determine the Position of the center of mass of a hemisphere of radius R

Ans. Let ρ be the density of the material of the hemisphere. Take its center O as the origin. The hemisphere can be assumed to be made of up large number of co-axial discs. Consider one such elementry disc of radius y and thickness dx at a distance x from the origin.

Mass of elementary disc dm = volume $\times$ density

$= \pi y^2 . dx \times \rho = \pi(R^2 - x^2) dx \cdot \rho$.

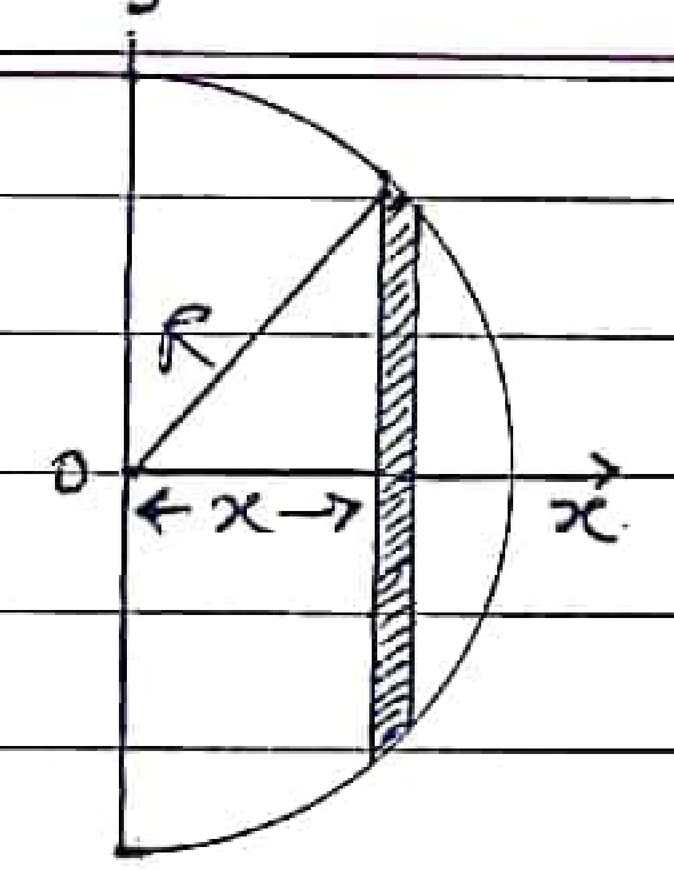
then Co-ordinate of CM is

$$x_{cm} = \frac{1}{m}\int x . dm = \frac{1}{m}\int_0^R x . \pi(R^2 - x^2)\rho . dx .$$

$= \frac{\pi\rho}{m}\left(\int_0^R (R^2x - x^3)dx\right) = \frac{\pi\rho}{m}\left[\frac{R^2x^2}{2} - \frac{x^4}{4}\right]_0^R$

$= \frac{\pi\rho}{m}\left[\frac{R^4}{2} - \frac{R^4}{4}\right] = \frac{\pi\rho}{m}\left[\frac{R^4}{4}\right]$

$= \frac{\pi\rho}{\frac{2}{3}\pi R^3 \rho} \cdot \frac{R^4}{4} = \frac{3}{8}R.$

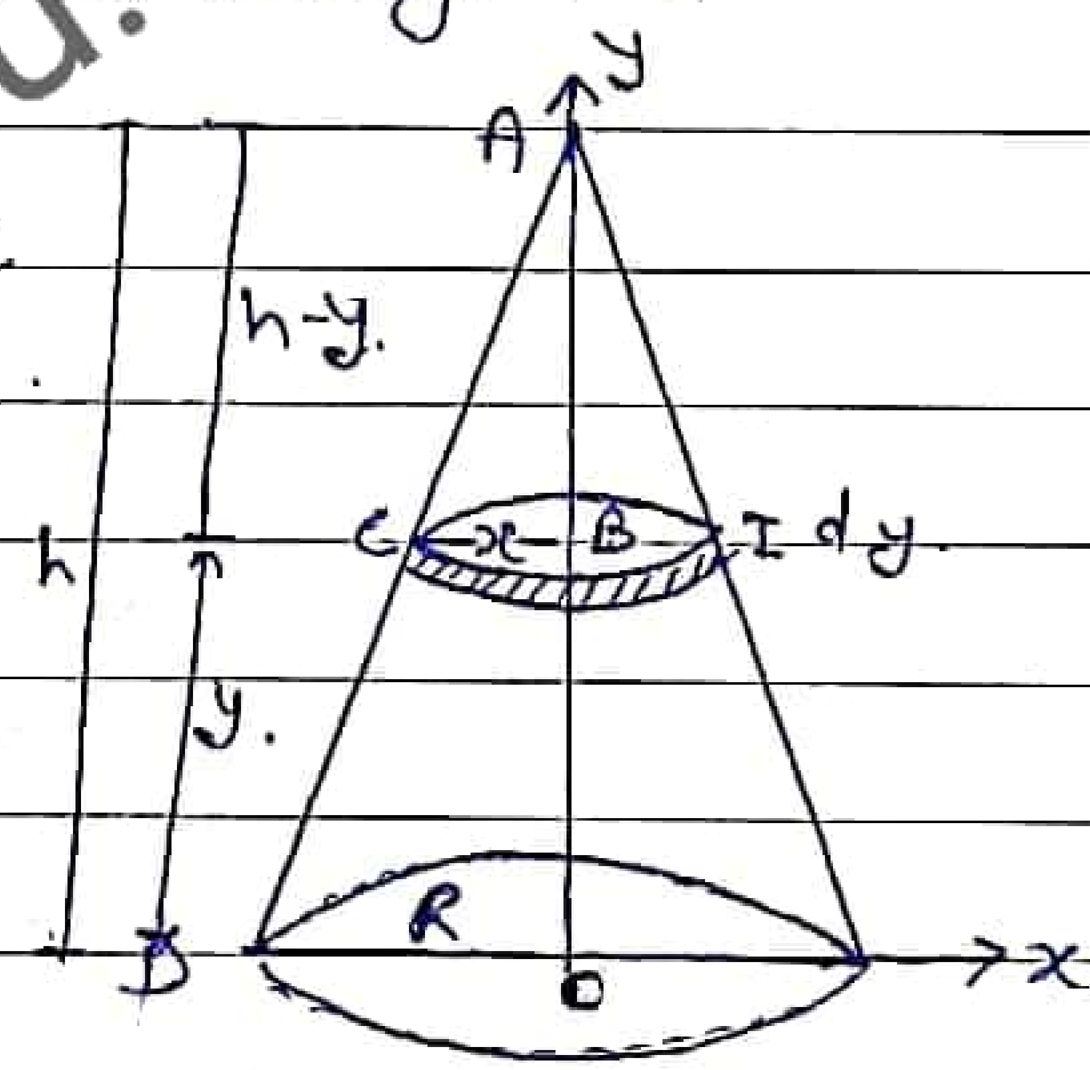


Simil. $y_{cm} = \int y.dm = 0$ and $z_{cm} = \int z\,dm = 0.$

Hence coordinate of center of mass of hemisphere $\left(\frac{3}{8}R, 0, 0\right)$

Q. Determine the Co-ordinates of the center of mass of a right circular solid cone of base radius R and height h.

Ans. Let ρ be the density of cone.
Take center of base O as the origin. The cone can be assumed to be made up of a large number of circular discs of different radius and mass.
Let one disc of radius x and mass dm at a height y from the base.



In similar triangle AOD and ABC.

$\frac{AO}{AB} = \frac{AD}{AC} = \frac{OD}{BC}$

$\frac{h}{h-y} = \frac{R}{x}$ then $x = \frac{R(h-y)}{h}.$

mass of disc dm = volume x density.

$= \pi x^2 dy \times \rho = \pi\rho dy \cdot \frac{R^2(h-y)^2}{h^2}$

the co-ordinate of center of mass.

$x_{cm} = \frac{1}{m}\int x\,dm = 0.$

$$y_{cm} = \frac{1}{m}\int y \cdot dm$$

$$= \frac{1}{m}\int_0^h y \cdot \pi\rho \frac{R^2(h-y)^2}{h^2}\, dy$$

$$= \frac{\pi\rho R^2}{mh^2}\int_0^h y(h-y)^2 \cdot dy$$

$$= \frac{\pi\rho R^2}{mh^2}\int_0^h y(h^2+y^2-2hy)\,dy$$

$$= \frac{\pi\rho R^2}{mh^2}\int_0^h (yh^2+y^3-2hy^2)\,dy$$

$$= \frac{\pi\rho R^2}{mh^2}\left[\frac{y^2h^2}{2}+\frac{y^4}{4}-2\cdot\frac{hy^3}{3}\right]_0^h$$

$$= \frac{\pi\rho R^2}{mh^2}\left[\frac{h^4}{2}+\frac{h^4}{4}-\frac{2h^4}{3}\right]$$

$$= \frac{\pi\rho R^2 h^4}{mh^2}\left[\frac{6+3-8}{12}\right]$$

$$= \frac{\pi\rho R^2 h^2}{m} \times \frac{1}{12} = \frac{\cancel{\pi\rho R^2} h^{\cancel{2}}}{4 \cdot \cancel{12}} \cdot \frac{1}{\frac{1}{\cancel{3}}\cancel{\pi R^2 h\rho}}$$

$$= \frac{h}{4}$$

Sim. $z_{cm} = \frac{1}{m}\int z\, dm = 0$

Co-ordinate $= \left(0, \frac{h}{4}, 0\right)$.