Fluid Mechanics :-

⇒ **Fluid :-** Fluid mechanics deals with the behaviour of fluids at rest and in motion. Fluids comprise the liquid and gas (or vapour) phases of the physical forms in which matter exists. The distinction between a fluid and the solid state of matter is clear if you compare fluid and solid behaviour.

⇒ **Ideal Fluid :-** An ideal liquid is incompressible and non- viscous in nature. An incompressible liquid means the density of the liquid is constant, it is independent of the variations in pressure. A non- viscous liquid means that, parts of the liquid in contact do not exert any tangential force on each other. Thus, there is no friction between the adjacent layers of a liquid. The force by one part of the liquid on the other part is perpendicular to the surface of contact.

Density of a liquid :-

Density (ρ) of any substance is defined as the mass per unit volume or

$$\rho = \frac{\text{mass}}{\text{volume}} \quad \text{or} \quad \boxed{\rho = \frac{m}{V}}$$

⇒ Relative Density (RD):- It is the ratio of density of the substance to the density of water at 4°C. Hence,

$$RD = \frac{\text{Density of Substance}}{\text{Density of water at } 4^\circ C}$$

⇒ Density of a Mixture of two or more liquids:-

Case-1:- Suppose two liquids of densities ρ_1 and ρ_2 having masses m_1 and m_2 are mixed together. Then, the density of the mixture will be

$$\rho = \frac{\text{Total mass}}{\text{Total Volume}} = \frac{(m_1 + m_2)}{(V_1 + V_2)}$$

$$= \frac{(m_1+m_2)}{\left(\frac{m_1}{\rho_1}+\frac{m_2}{\rho_2}\right)}$$

If $m_1 = m_2$, then $\boxed{\rho = \frac{2\rho_1\rho_2}{\rho_1+\rho_2}}$

Case-2:- If two liquids of densities ρ_1 and ρ_2 having volumes V_1 and V_2 are mixed, then the density of the mixture is,

$$\rho = \frac{\text{Total mass}}{\text{Total volume}} = \frac{m_1+m_2}{V_1+V_2}$$

$$= \frac{\rho_1 V_1 + \rho_2 V_2}{V_1+V_2}$$

If $V_1 = V_2$, then $\boxed{\rho = \frac{\rho_1+\rho_2}{2}}$

⇒ Effect of Temperature on Density:-

As the temperature of a liquid is increased, the mass remains the same while the volume is increased and hence, the density of the liquid decreases $\left(\text{as } \rho \propto \frac{1}{V}\right)$. Thus,

$$\frac{\rho'}{\rho} = \frac{V}{V'} = \frac{V}{V+dV} = \frac{V}{V+V\gamma\Delta\theta}$$

or
$$\frac{\rho'}{\rho} = \frac{1}{1+\gamma\Delta\theta}$$

Here, γ = thermal coefficient of volume expansion
and $\Delta\theta$ = rise in temperature

$$\therefore \quad \boxed{\rho' = \frac{\rho}{1+\gamma\Delta\theta}}$$

We can also write $\rho' = \rho(1+\gamma\Delta\theta)^{-1}$
or $\boxed{\rho' \simeq \rho(1-\gamma\Delta\theta)}$ $\quad$ (If $\gamma\Delta\theta << 1$)

⇒ Effect of Pressure on Density :-

As pressure is increased, volume decreases and hence density will increase. Thus,

$$\rho \propto \frac{1}{V}$$

$$\therefore \quad \frac{\rho'}{\rho} = \frac{V}{V'} = \frac{V}{V+dV} = \frac{V}{V-\left(\frac{dp}{B}\right)V}$$

or $$\frac{\rho'}{\rho} = \frac{1}{1-\frac{dp}{B}}$$

Here, dp = Change in pressure
and B = bulk modulus of elasticity of the liquid
Therefore, $$\boxed{\rho' = \frac{\rho}{1-\frac{dp}{B}}}$$

$$\Rightarrow \quad \rho' = \rho\left(1-\frac{dp}{B}\right)^{-1}$$

or $$\rho' \approx \rho\left(1+\frac{dp}{B}\right) \qquad (\text{if } dp << B)$$

Further, $$\boxed{\rho'-\rho = \Delta\rho = \frac{\rho(dp)}{B} \text{ or } \frac{\rho(\Delta p)}{B}}$$

Eg:- Relative density of an oil is 0.8. Find the absolute density of oil in CGS and SI units.

Soln:- Density of oil (in CGS) = (RD) g/cm^3

$$= 0.8\ g/cm^3$$

$$= 800\ kg/m^3 \Rightarrow \text{(Ans)}$$

Pressure in a fluid :-

When a fluid (either liquid or gas) is at rest, it exerts a force perpendicular to any surface in contact with it, such as a container wall or a body immersed in the fluid. While the fluid as a whole is at rest, the molecules that makes up the fluid are in motion, the force exerted by the fluid is due to molecules colliding with their surroundings.

The pressure p is defined at that point as the normal force per unit area, i.e

$$\boxed{p = \frac{dF_\perp}{dA}} \qquad \left(\text{as } p = \frac{F}{A}\right)$$

$$\left[p = \frac{F_\perp}{A} \right]$$

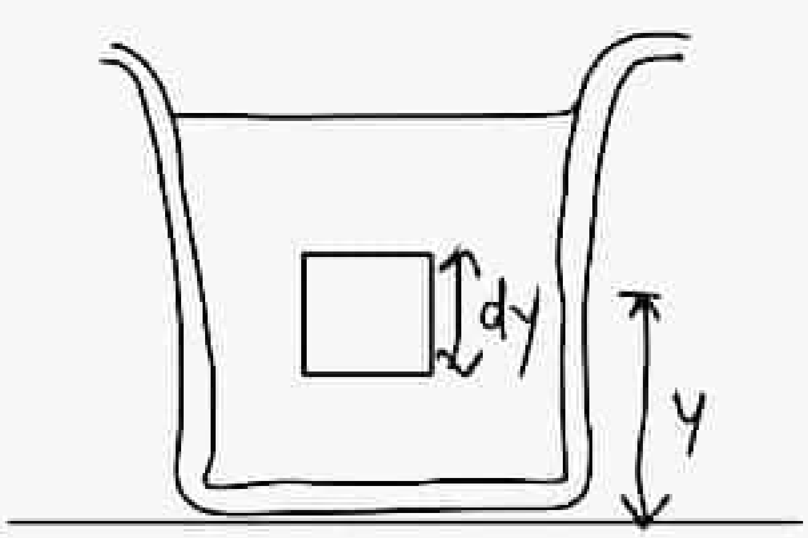
Atmospheric Pressure (P_0):-

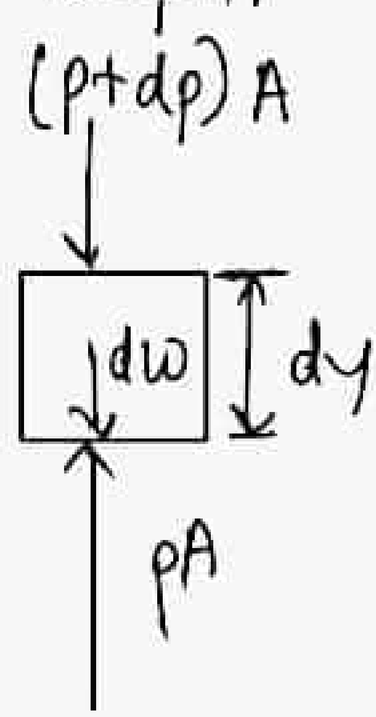
It is pressure of the earth's atmosphere. This changes with weather and elevation. Normal atmospheric pressure at sea level (an average value) is 1.013×10^5 Pa. Thus,

$$1 \text{ atm} = 1.013 \times 10^5 \text{ Pa}$$

⇒ Variation in Pressure with Depth:-

If the weight of the fluid can be neglected, the pressure in a fluid is the same throughout its volume. But often the fluid's weight is not negligible and under such condition pressure increases with increasing depth below the surface





$$dw = (\text{volume})(\text{density})(g) = (A\,dy)(\rho)(g) \text{ or}$$

$$dw = \rho g A\,dy$$

$$\Sigma F_y = 0$$

$\therefore$

$$pA - (p + dp)A - \rho g A\,dy = 0$$

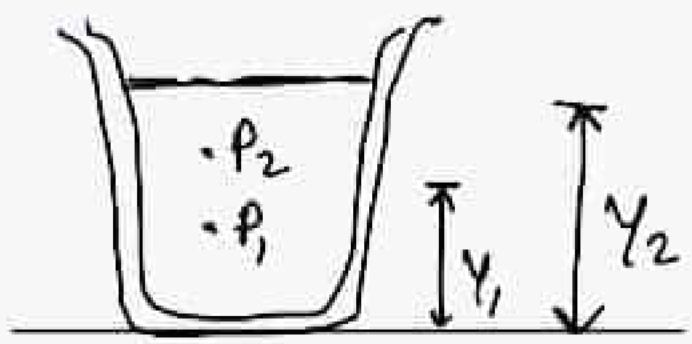
or

$$\boxed{\frac{dp}{dy} = -\rho g} \quad \text{——— (1)}$$

$$\int_{p_1}^{p_2} dp = -\rho g \int_{y_1}^{y_2} dy$$

or

$$\boxed{p_2 - p_1 = -\rho g (y_2 - y_1)} \quad \text{——— (2)}$$



$$h = y_2 - y_1$$

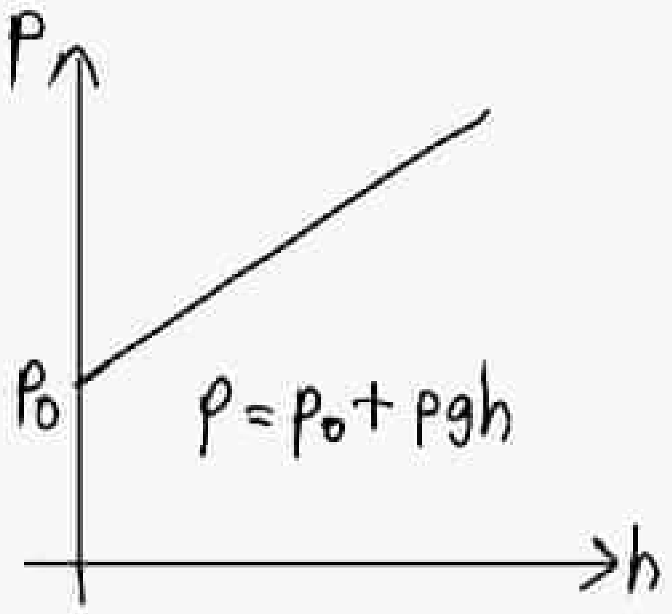
and eq^n (2) becomes

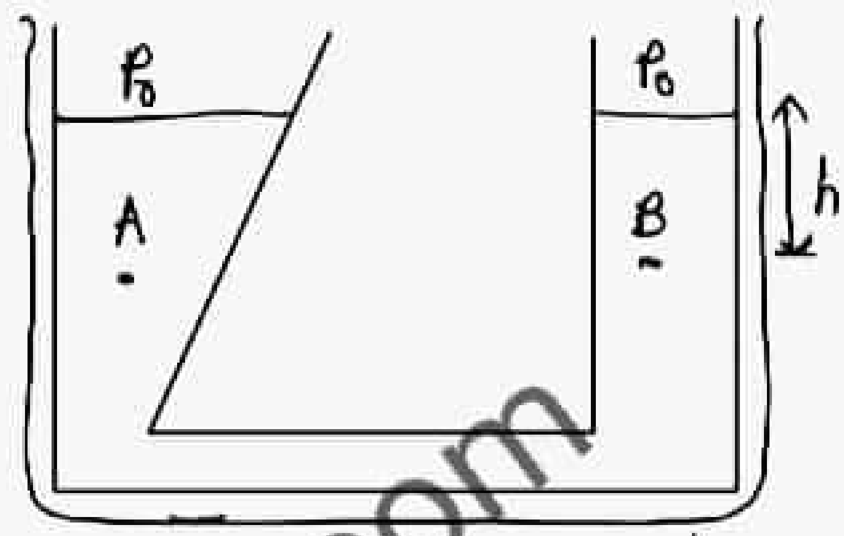
$$p_0 - p = -\rho g (y_2 - y_1) = -\rho g h$$

$\therefore$

$$\boxed{p = p_0 + \rho g h} \quad \text{——— (3)}$$

Thus, pressure increases linearly with depth, if ρ and g are uniform. A graph b/w p and h is

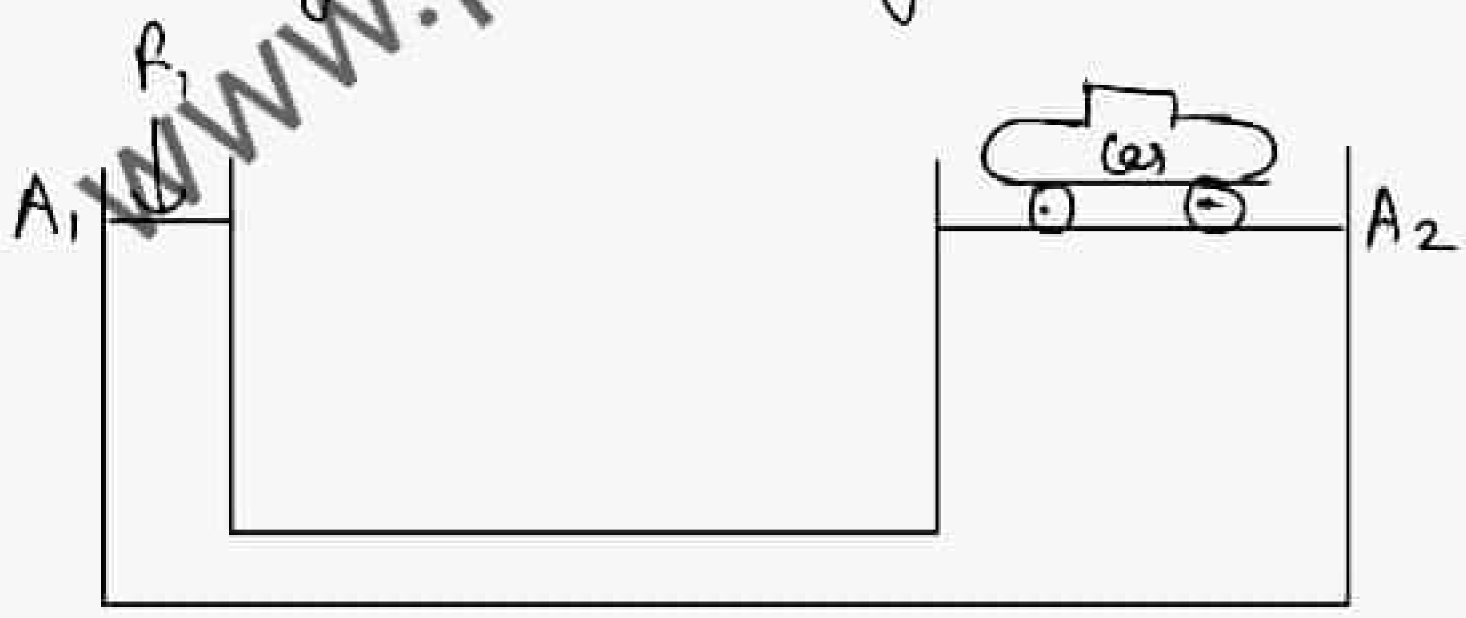




$$P_A = P_B = P_0 + \rho g h$$

Pascal's law:-

Pressure exerted anywhere in a confined incompressible fluid is transmitted equally in all directions throughout the fluid.

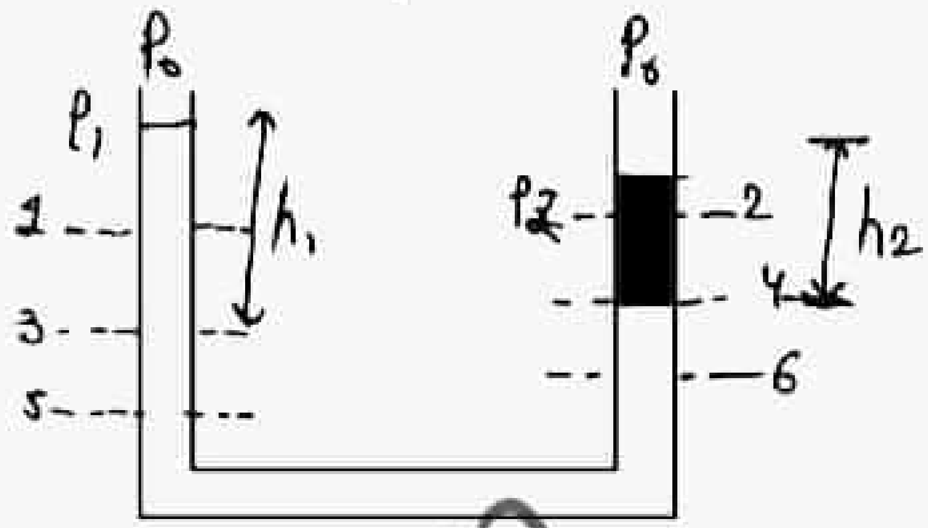


$$p = \frac{F_1}{A_1} = \frac{F_2}{A_2}$$

or $$F_2 = \frac{A_2}{A_1} \cdot F_1$$

Note:- In the same liquid pressure will be same at all points at the same level (provided their speeds are same).

for Eg:- , in the fig.



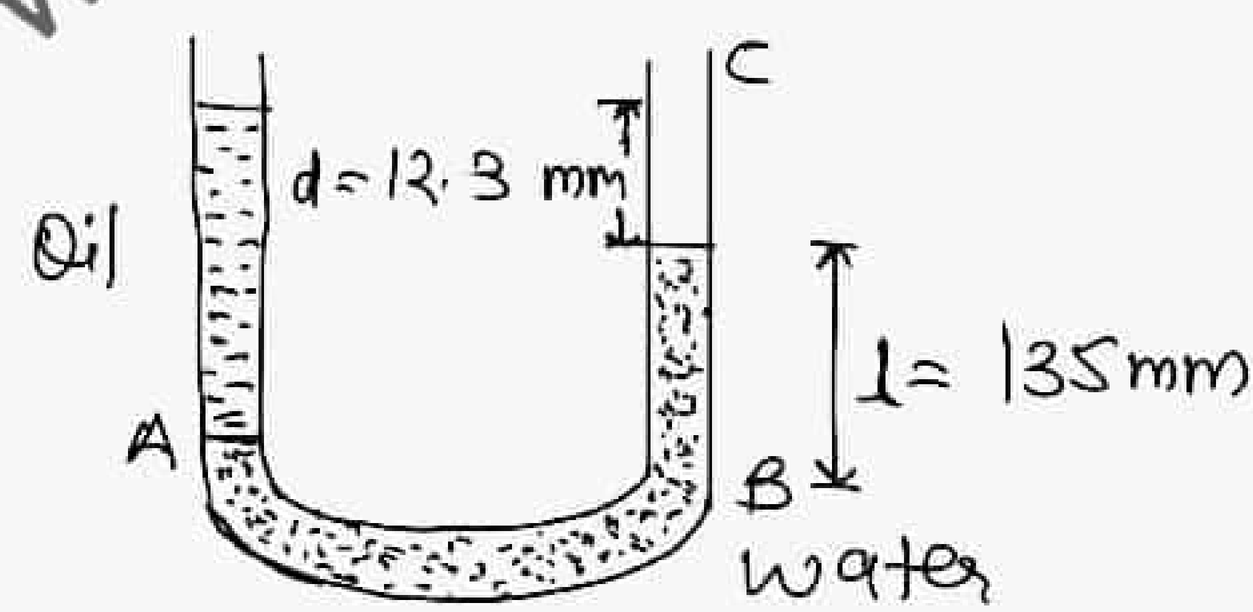
$P_1 \neq P_2$, $P_3 = P_4$ and $P_5 = P_6$

further $\quad P_3 = P_4$

$$\therefore \quad P_0 + \rho_1 g h_1 = P_0 + \rho_2 g h_2$$

or $\quad \rho_1 h_1 = \rho_2 h_2$ or $\boxed{h \propto \frac{1}{\rho}}$

Eg:- For the arrangement shown in the fig., What is the density of oil?

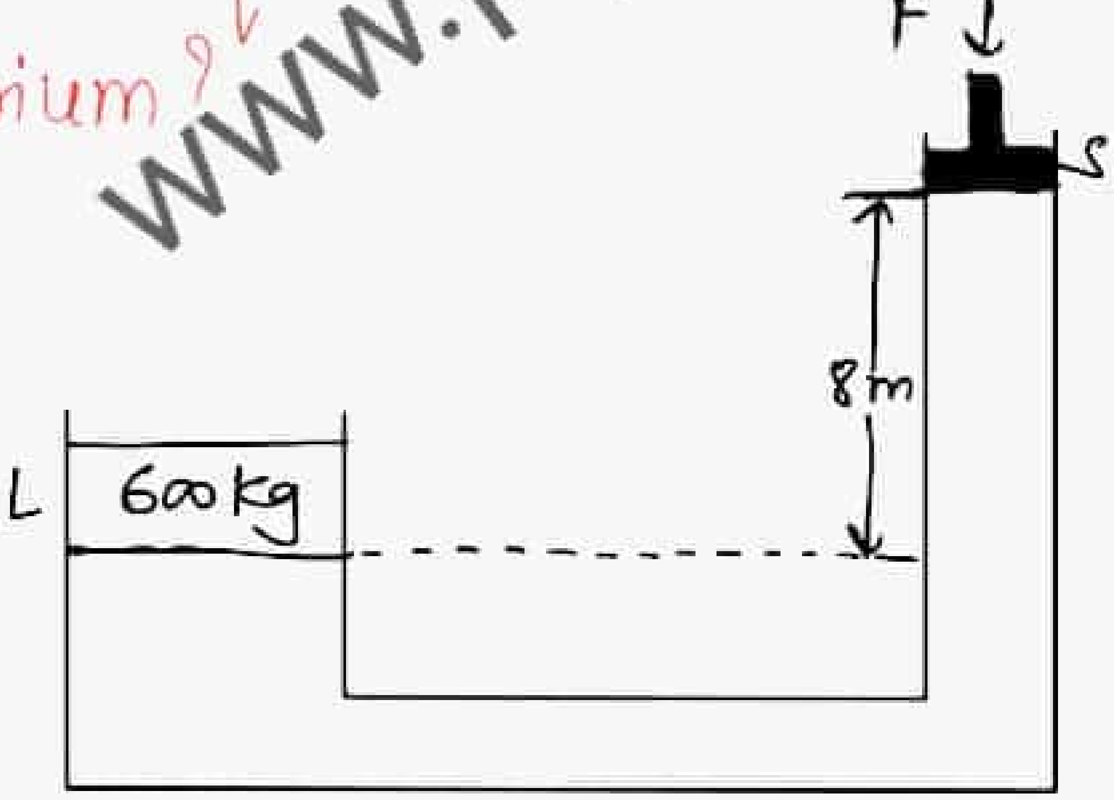


Sol^n:- $\qquad P_B = P_A$

$$\therefore \quad P_0 + \rho_w g L = P_0 + \rho_{oil}(L+d)g$$

$$\Rightarrow \rho_{oil} = \frac{\rho_w L}{L+d} = \frac{(1000)(135)}{(135+12.3)} = 916 \text{ kg/m}^3 \text{ (Ans)}$$

Eg:- For the system shown in fig., the cylinder on the left, at L, has a mass of 600 kg and a cross-sectional area of 800 cm^2. The piston on the right, at S, has cross sectional area 25 cm^2 and negligible weight. If the apparatus is filled with oil ($\rho = 0.78$ g/cm^3), what is the force F required to hold the system in equilibrium?



Sol^n:- Let A_1 = area of cross-section on LHS and
A_2 = area of cross-section on RHS

Equating the pressure on two sides of the dotted line. Then,

$$\frac{f}{A_2} + \rho g h = \frac{Mg}{A_1} \qquad (M = 600\ kg)$$

$$\therefore \quad f = A_2\left[\frac{Mg}{A_1} - \rho g h\right]$$

$$= (25 \times 10^{-4})\left[\frac{600 \times 9.8}{800 \times 10^{-4}} - 780 \times 9.8 \times 8\right]$$

$$= 31\ N$$

Pressure Difference in Accelerating Fluids :-

If fluids are at rest then pressure does not change in horizontal direction.
It changes only in vertical direction.
At a height difference 'h', difference in pressure (or change in pressure) is

$$\boxed{\Delta p = \pm \rho g h}$$

Pressure increases with depth. So, $\Delta p = +\rho g h$ in moving downwards and $\Delta p = -\rho g h$ in moving upwards

If fluids are accelerated then pressure changes in both horizontal and vertical directions.

⇒ In Horizontal Direction :-

(i) Pressure decreases in the direction of acceleration.

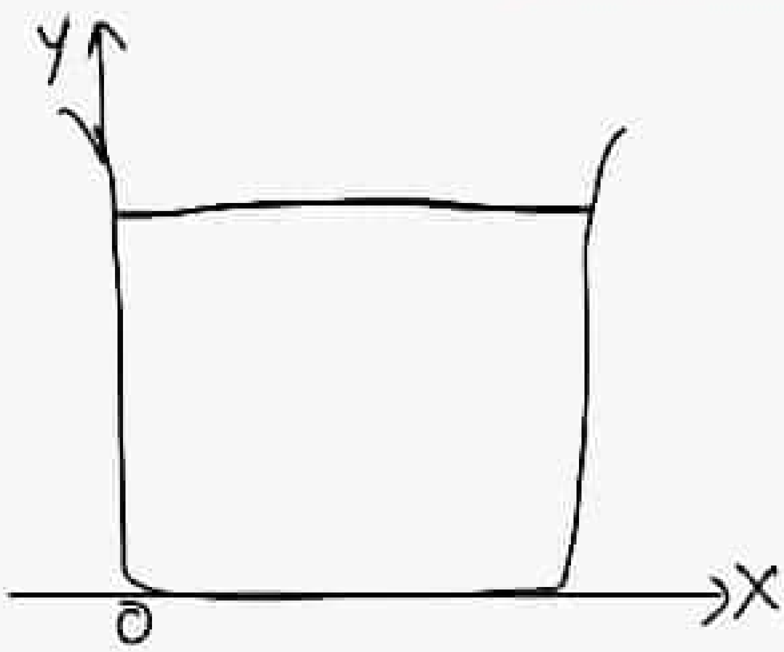
(ii) At a horizontal distance x change in pressure is

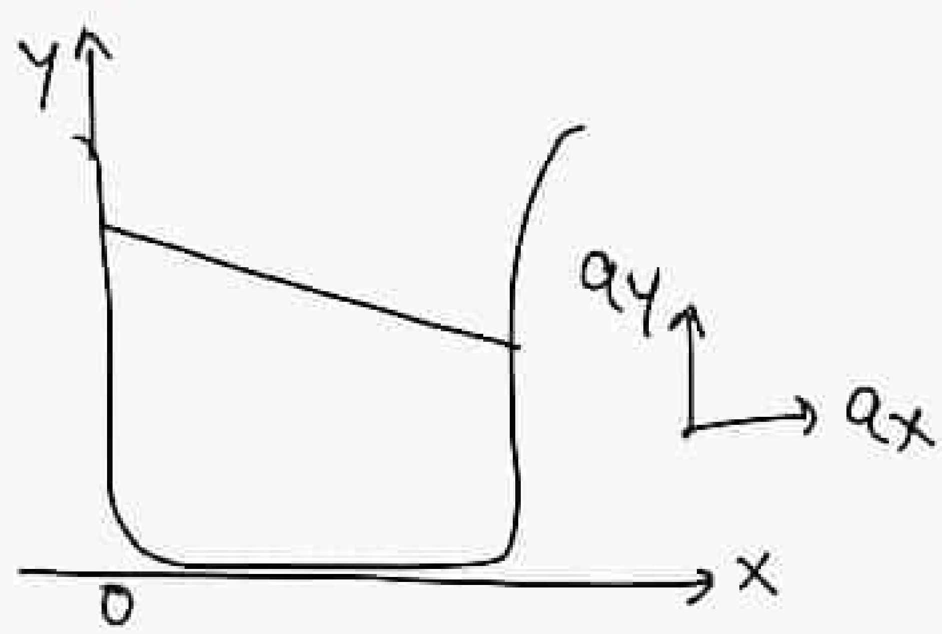
$$\boxed{\Delta p = \pm \rho a x}$$

⇒ In Vertical Direction :-

Instead of 'g' in the eq^n, $\Delta p = \pm \rho g h$ take effective value of g or g_e. Thus,

$$\boxed{\Delta p = \pm \rho g_e h}$$



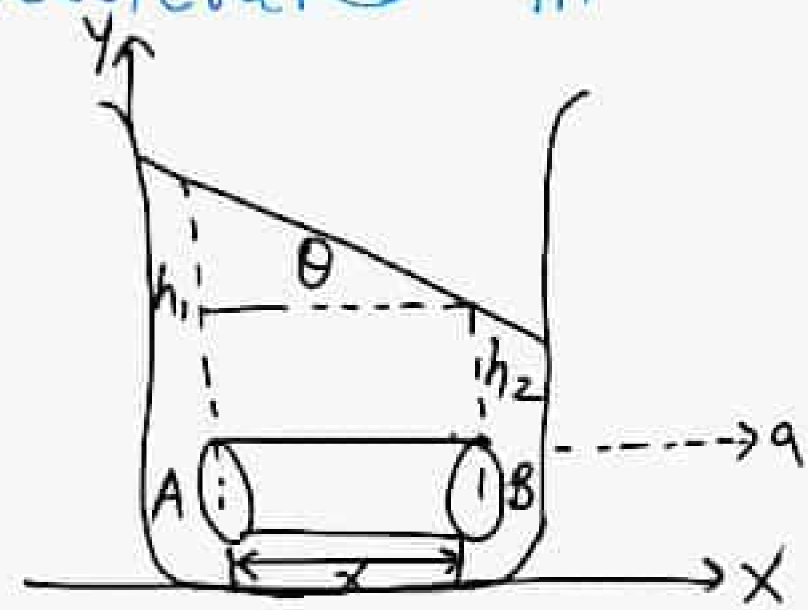


$$\frac{dp}{dx} = 0 \quad \text{and} \quad \frac{dp}{dy} = -\rho g$$

Suppose the beaker is accelerated and it has components of acceleration a_x and a_y in x and y directions respectively, then the pressure decreases along both x and y- directions.

$$\boxed{\frac{dp}{dx} = -\rho a_x \quad \text{and} \quad \frac{dp}{dy} = -\rho(g+a_y)}$$

⇒ Free Surface of a liquid Accelerated in Horizontal Direction:-



Consider a liquid placed in a beaker which is accelerating horizontally with an acceleration 'a'. Let 'A' and 'B' be two points in the liquid at a separation x in the same horizontal line. As we have seen in this case.

$$\frac{dp}{dx} = -\rho a$$

or $\quad dp = -\rho a \, dx$

Integrating this with proper limits, we get

$$p_A - p_B = \rho a x \quad \text{———} \quad (3)$$

Further, $\quad p_A = p_0 + \rho g h_1$

and $\quad p_B = p_0 + \rho g h_2$

Substituting in Eq^n (3), we get

$$\rho h (h_1 - h_2) = \rho a x$$

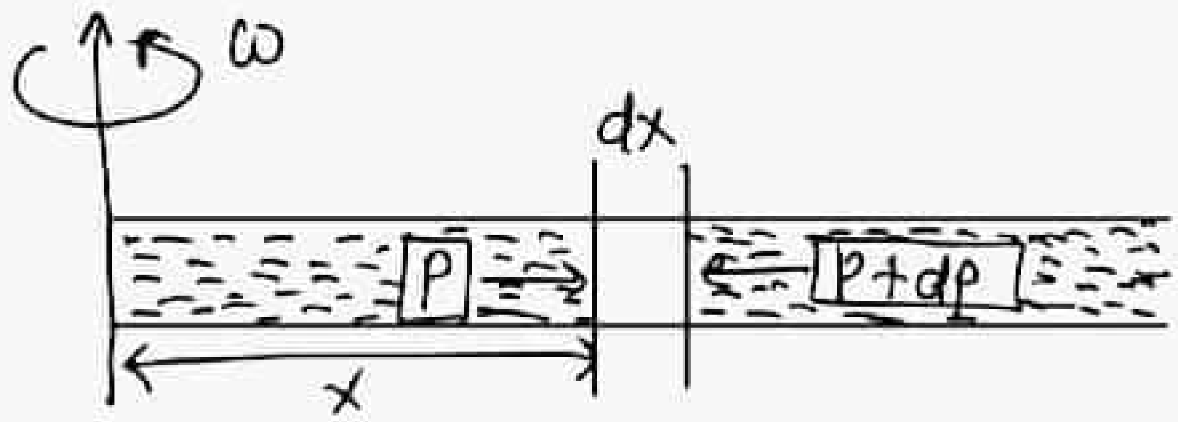
$$\therefore \quad \frac{h_1 - h_2}{x} = \frac{a}{g} = \tan\theta$$

$$\therefore \quad \boxed{\tan\theta = \frac{a}{g}}$$

Pressure Difference in Rotating Fluids:-

In a rotating fluid (also accelerating) pressure increases in moving away from the rotational axis. At a distance 'x' from the rotational axis, pressure difference is -

$$\boxed{\Delta p = \pm \frac{\rho \omega^2 x^2}{2}}$$



$$dm = (\text{density})(\text{volume})$$

or $$dm = (\rho A dx)$$

$$a = x\omega^2 \quad (\text{as } a = R\omega^2)$$

$$(p+dp)A - (p)A = (dm)a = (\rho A dx)(x\omega^2)$$

$$dp = (\rho x \omega^2)\, dx$$

$$\Delta p = \int_0^x (\rho \omega^2) x \, dx$$

or $$\Delta p = \frac{\rho \omega^2 x^2}{2}$$

Hence proved

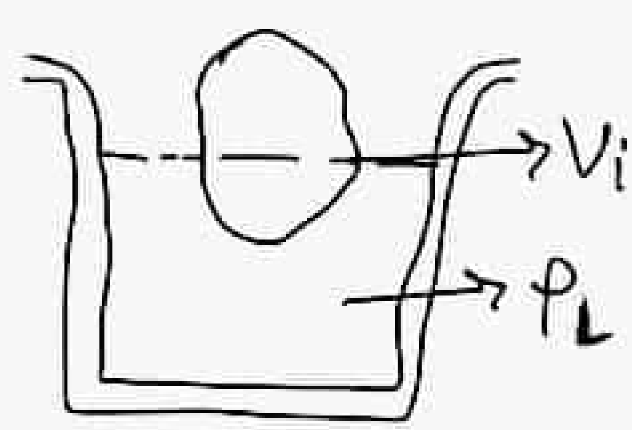
Archimedes' Principle :-

A body wholly or partially submerged in a fluid is buoyed up by a force equal to the weight of the displaced fluid.
This result is known as Archimedes' principle.
The magnitude of buoyant force (F) is given by,

$$\boxed{F = V_i \rho_L g}$$

Here, V_i = immersed volume of solid, ρ_L = density of liquid
and g = acceleration due to gravity

Law of Floatation :-



Consider an object of volume V and density ρ_s floating in a liquid of density ρ_L. Let V_i be the volume of object immersed in the liquid. For equilibrium of object,

$\therefore$ Weight = Upthrust

$$V \rho_s g = V_i \rho_L g$$

$$\therefore \frac{V_i}{V} = \frac{\rho_s}{\rho_L} \qquad \text{— (1)}$$

This is the fraction of Volume immersed in liquid.

Percentage of volume immersed in liquid $= \frac{V_i}{V} \times 100 = \frac{\rho_s}{\rho_L} \times 100$

(i) If $\rho_s < \rho_L$, only fraction of body will be immersed in the liquid. This fraction will be given by the above eqn.

(ii) If $\rho_s = \rho_L$, the whole of the rigid body will be Immersed in the liquid. Hence, the body remains floating in the liquid wherever it is left.

(iii) If $\rho_s > \rho_L$, the body will sink.

Buoyant force in Accelerating fluids:-

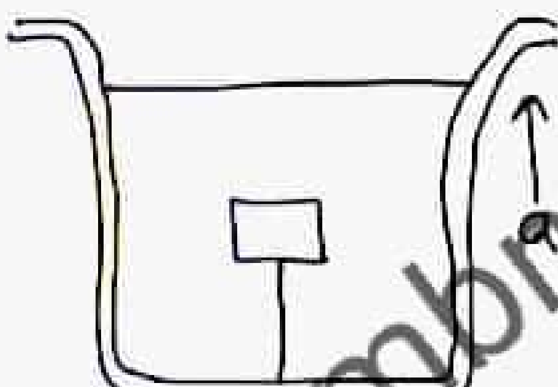
Suppose a body is dipped inside a liquid of density ρ_L placed in an elevator moving with an acceleration a. The buoyant force F in this case becomes,

$$\boxed{F = V\rho_l g_{eff}}$$

(V = immersed volume of solid or V_i)

$$g_{eff} = |g - a|$$

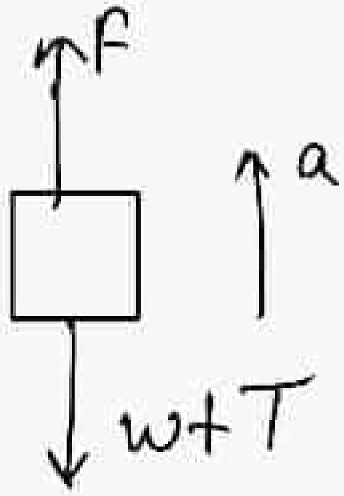
Eg:- A block of mass 1 kg and density 0.8 g/cm^3 is held stationary with the help of a string as shown in fig. The tank is accelerating vertically upwards with an acceleration $a = 1.0$ m/s^2. Find



(a) the tension in the string,

(b) If the string is now cut find the acceleration of block. (Take $g = 10$ m/s^2 and density of water $= 10^3$ kg/m^3)

Solⁿ:- (a)



f = upthrust force

$= V\rho_w (g+a)$

$= \left(\dfrac{\text{mass of block}}{\text{density of block}}\right) \rho_w (g+a)$

$= \left(\dfrac{1}{800}\right)(2000)(10+1) = 13.75 \text{ N}$

$w = mg = 10 \text{ N}$

Eq^n of motion of the block is,

$f - T - w = ma$

$\therefore \quad 13.75 - T - 10 = 1 \times 1$

$\therefore \quad T = 2.75 \text{ N} \quad \rightarrow$ (Ans)

(b) When the string is cut, $T = 0$

$\therefore \quad a = \dfrac{f - w}{m}$

$= \dfrac{13.75 - 10}{1}$

$= 3.75 \text{ m/s}^2$

flow of fluids:-

⇒ Steady Flow:- If the velocity of fluid particles at any point does not vary with time, the flow is said to be steady. Steady flow is also called streamlined or laminar flow. The velocity at different points may be different. Hence, in the fig,

$V_1 =$ Constant, $V_2 =$ Constant, $V_3 =$ Constant but

$V_1 \neq V_2 \neq V_3$

⇒ Principle of Continuity:-

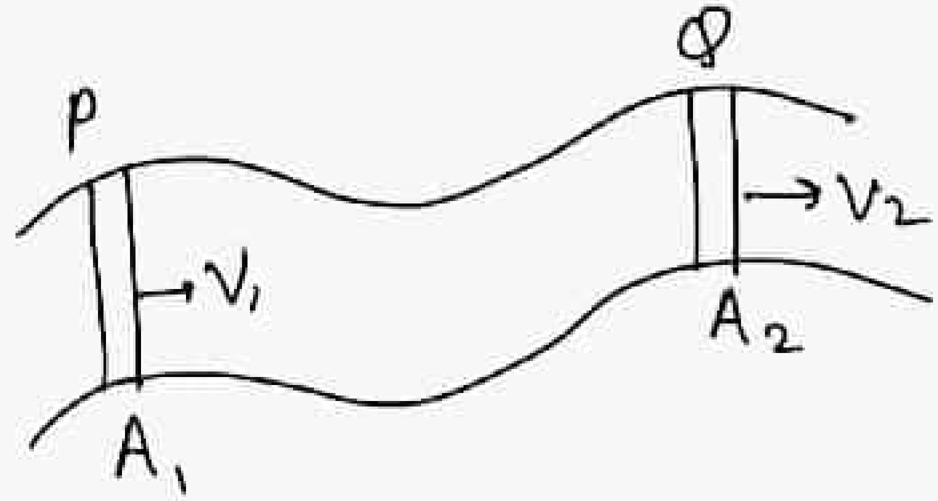
It states that, when an incompressible and non-viscous liquid flows in a stream lined motion through a tube of non-uniform cross-section, then the product of the area of cross section and the velocity of flow is same at every point in the tube.

Thus,

$$\boxed{A_1 V_1 = A_2 V_2} \quad \text{or}$$

$$Av = \text{Constant} \quad \text{or}$$

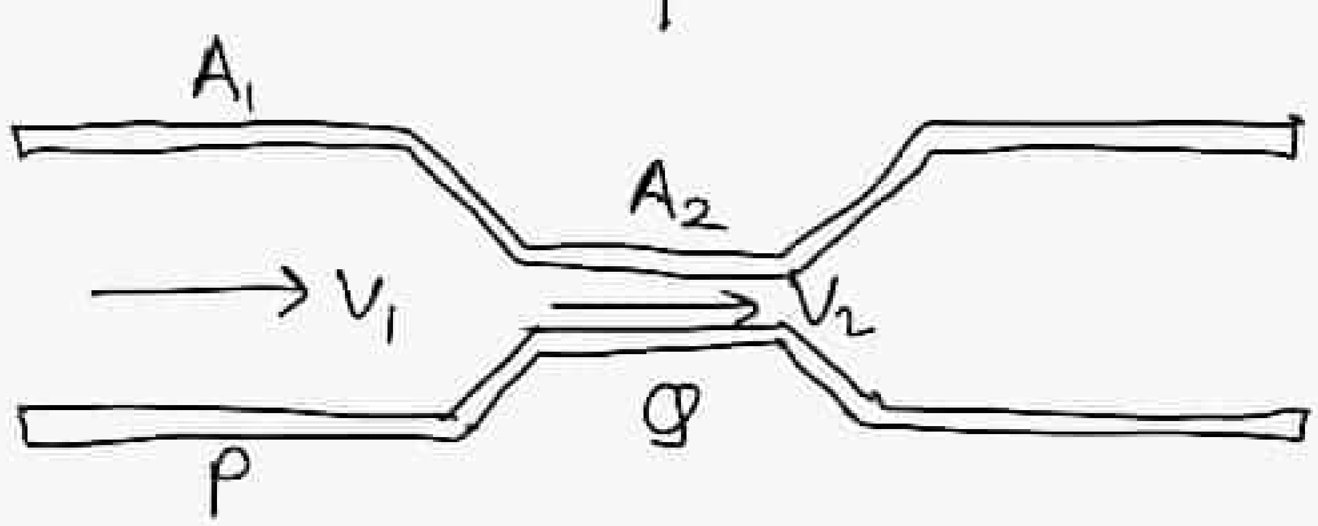
$$v \propto \frac{1}{A}$$



This is basically the law of Conservation of mass in fluid dynamics

⇒ Proof :-

Consider two cross sections P and Q of area A_1 and A_2 of a tube through which a fluid is flowing. Let v_1 and v_2 be the speeds at these two cross sections. Then, being an incompressible fluid, mass of fluid going through P in a time interval Δt = mass of fluid passing through Q in the same interval of time Δt.



$\therefore \quad A_1 v_1 \rho \Delta t = A_2 v_2 \rho \Delta t$

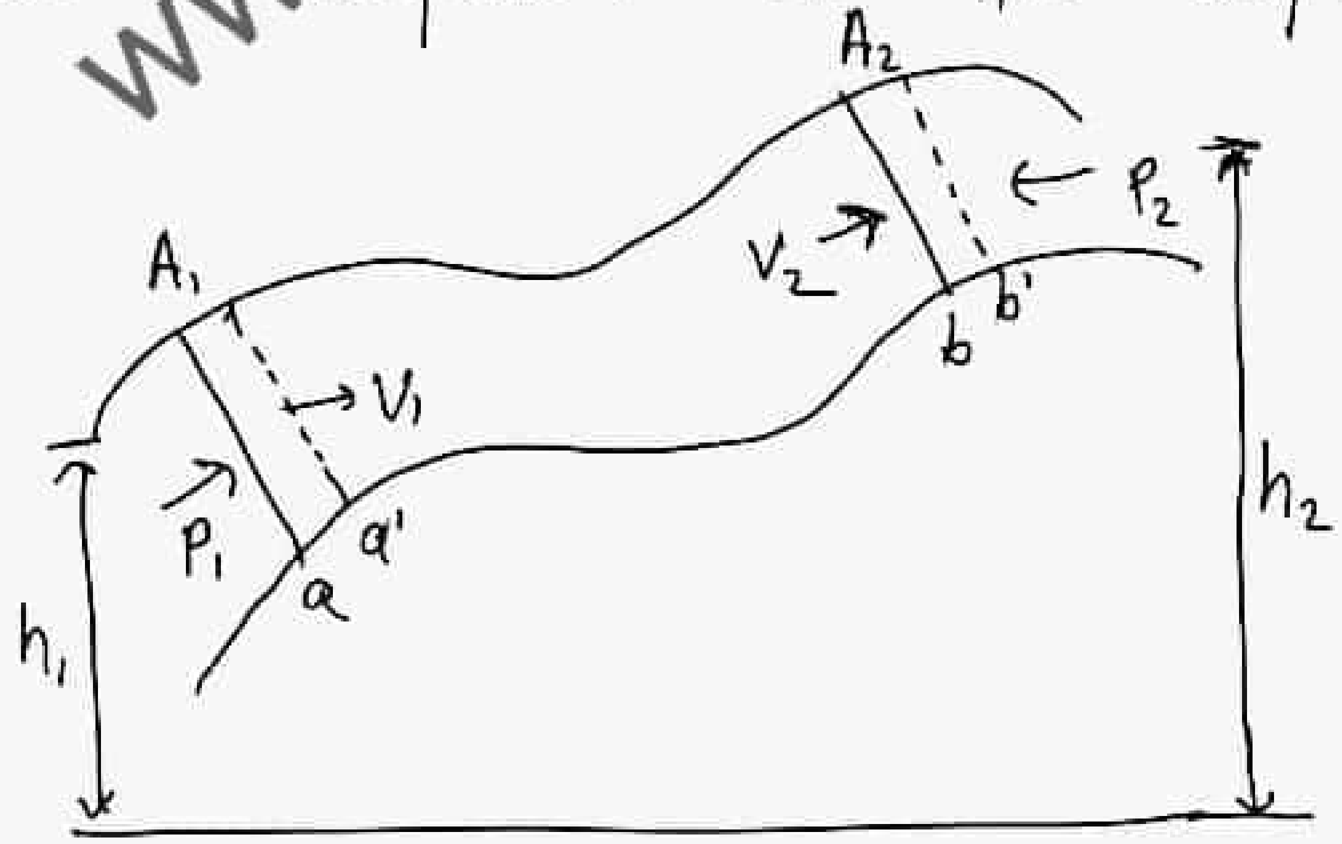
or $\quad A_1 v_1 = A_2 v_2$ (Proved)

Therefore, the velocity of the liquid is smaller in the wider parts of the tube and larger in the narrower parts.

or $\quad v_2 > v_1 \quad$ as $\quad A_2 < A_1$

Bernoulli's Equation:-

Bernoulli's eqn relates the pressure, flow speed and height for flow of an ideal (incompressible and non-viscous) fluid. The pressure of a fluid depends on height as in the static situation and it also depends on the speed of flow.



When an incompressible fluid flows along a tube with varying cross section, its speed must change, and so, an element of fluid must have an acceleration. If the tube is horizontal, the force that causes this acceleration has to be applied by the surrounding fluid. This means that the pressure must be different in regions of different cross section. When a horizontal flow tube narrows and a fluid element speeds up, it must be moving towards a region of lower pressure in order to have a net forward force to accelerate it. If the elevation also changes, this causes additional pressure difference.

To derive Bernoulli's eq^n, we apply the work-energy theorem to the fluid in a section of the fluid element. Consider the element of fluid that at some initial time lies between two cross sections a and b. The speeds at the lower and upper ends are v_1 and v_2. In a small time interval, the fluid that is initially at a moves to a' a distance $aa' = ds_1 = v_1\, dt$ and the fluid that is initially at b moves to b' a distance $bb' = ds_2 = v_2\, dt$. The cross-section areas at the two ends are A_1 and A_2 as shown. The fluid is incompressible, hence, by the continuity eqn, the volume of fluid dV passing through any cross-section during time dt is the same. That is,

$$\boxed{dV = A_1 ds_1 = A_2 ds_2}$$

⇒ Work Done on the Fluid Element:-

During time interval dt. The pressure at the two ends are p_1 and p_2, the force on the cross section at a is $p_1 A_1$ and the force at b is $p_2 A_2$. The net work done dW on the element by the surrounding fluid during this displacement is,

$$dW = p_1 A_1 ds_1 - p_2 A_2 ds_2 = (p_1 - p_2)\, dV \quad \text{—(1)}$$

The second term is negative, because the force at b opposes the displacement of the fluid. This work dw is due to forces other than the conservative force is gravity, so it equals the change in total mechanical energy (kinetic plus potential). The mechanical energy for the fluid between sections a and b does not change.

⇒ Change in Potential Energy:-

At the beginning of dt the potential energy for the mass between a and a' is $dmgh_1 = \rho dVgh_1$. At the end of dt the potential energy for the mass between b and b' is $dmgh_2 = \rho dVgh_2$. The net Change in potential energy dU during dt is,

$$dU = \rho(dV)g(h_2 - h_1) \quad \text{— (2)}$$

⇒ Change in Kinetic Energy:-

At the beginning of dt the fluid between a and a' has volume $A_1 ds_1$, mass $\rho A_1 ds_1$ and kinetic energy $\frac{1}{2}\rho(A_1 ds_1)v_1^2$. At the end of dt the fluid between b and b' has kinetic energy $\frac{1}{2}\rho(A_2 ds_2)v_2^2$. The net change in kinetic energy dK during time dt is,

$$dK = \frac{1}{2}\rho(dV)(v_2^2 - v_1^2) \quad \text{——} (3)$$

Combining Eq^{n}s (1), (2) and (3) in the energy eq^n,

$(dW = dK + dU)$

We obtain,

$$(p_1 - p_2)dV = \frac{1}{2}\rho dV(v_2^2 - v_1^2) + \rho(dV)g(h_2 - h_1)$$

or

$$p_1 - p_2 = \frac{1}{2}\rho(v_2^2 - v_1^2) + \rho g(h_2 - h_1) \quad \text{——}(4)$$

$$\boxed{p_1 + \rho g h_1 + \frac{1}{2}\rho v_1^2 = p_2 + \rho g h_2 + \frac{1}{2}\rho v_2^2}$$

(Bernoulli's Eq^n)

$$\left(p + \rho g h + \frac{1}{2}\rho v^2 = \text{Constant}\right)$$

Eg:- Water is flowing through a horizontal tube of non-uniform cross section. At a place, the radius of the tube is 1.0 cm and the velocity of water is 2m/s. What will be the velocity of water where the radius of the pipe is 2.0 cm?

Solⁿ:- Using eqⁿ of Continuity, $A_1 v_1 = A_2 v_2$

$$v_2 = \left(\frac{A_1}{A_2}\right) v_1$$

or $$v_2 = \left(\frac{\pi r_1^2}{\pi r_2^2}\right) v_1 = \left(\frac{r_1}{r_2}\right)^2 v_1$$

Substituting the values, we get

$$v_2 = \left(\frac{1.0 \times 10^{-2}}{2.0 \times 10^{-2}}\right)^2 (2)$$

or $v_2 = 0.5 \text{ m/s}$ $\Longrightarrow$ (Ans)

Eg:- Calculate the rate of flow of glycerine of density 1.25×10^3 kg/m³ through the conical section of a pipe, if the radii of its ends are 0.1 m and 0.04 m and the pressure drop across its length is $10\ N/m^2$

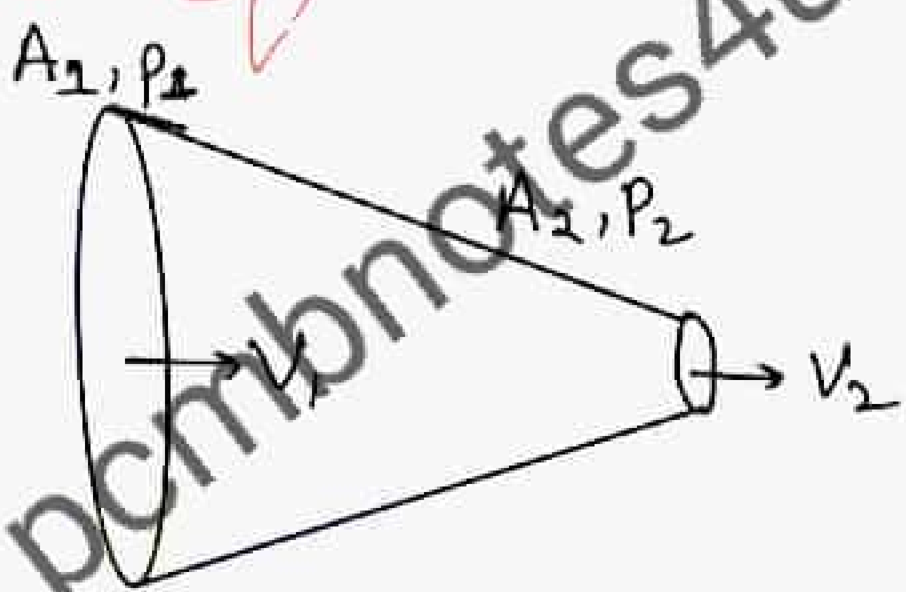
Solⁿ:- From continuity eqⁿ,

$$A_1 v_1 = A_2 v_2$$

or $$\frac{v_1}{v_2} = \frac{A_2}{A_1} = \frac{\pi r_2^2}{\pi r_1^2} = \left(\frac{r_2}{r_1}\right)^2$$

$$= \left(\frac{0.04}{0.1}\right)^2 = \frac{4}{25} \quad \text{--- (1)}$$

From Bernoulli's Eqⁿ,



$$P_1 + \tfrac{1}{2}\rho v_1^2 = P_2 + \tfrac{1}{2}\rho v_2^2 \quad \text{or} \quad v_2^2 - v_1^2 = \frac{2(P_1 - P_2)}{\rho}$$

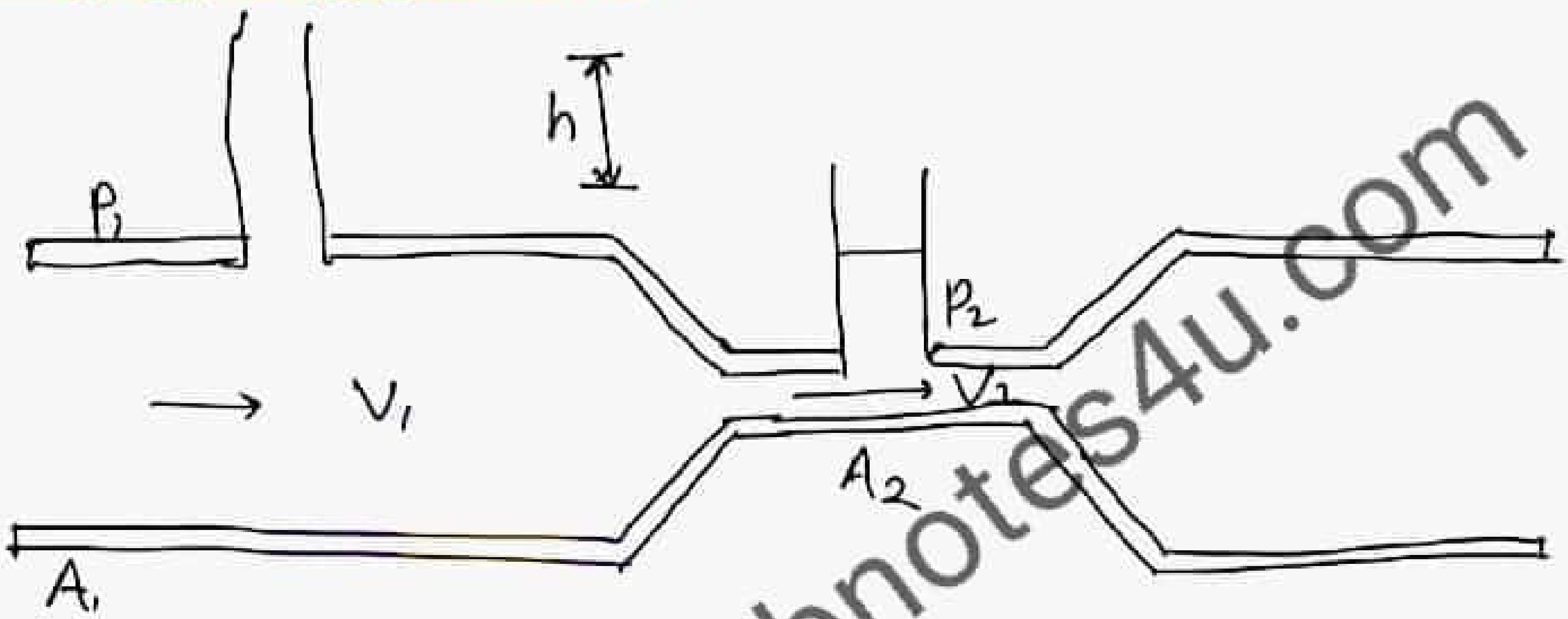
or $$v_2^2 - v_1^2 = \frac{2 \times 10}{1.25 \times 10^3} = 1.6 \times 10^{-2}\ m^2/s^2 \quad \text{--- (2)}$$

Solving Eqⁿs (1) and (2), we get $v_2 \approx 0.128\ m/s$

∴ Rate of volume flow through the tube

$$\mathcal{Q} = A_2 v_2 = (\pi r_2^2) v_2$$
$$= \pi (0.04)^2 (0.128)$$
$$= 6.43 \times 10^{-4}\ m^3/s$$

Venturimeter:-



$$P_1 + \tfrac{1}{2}\rho v_1^2 = P_2 + \tfrac{1}{2}\rho v_2^2$$

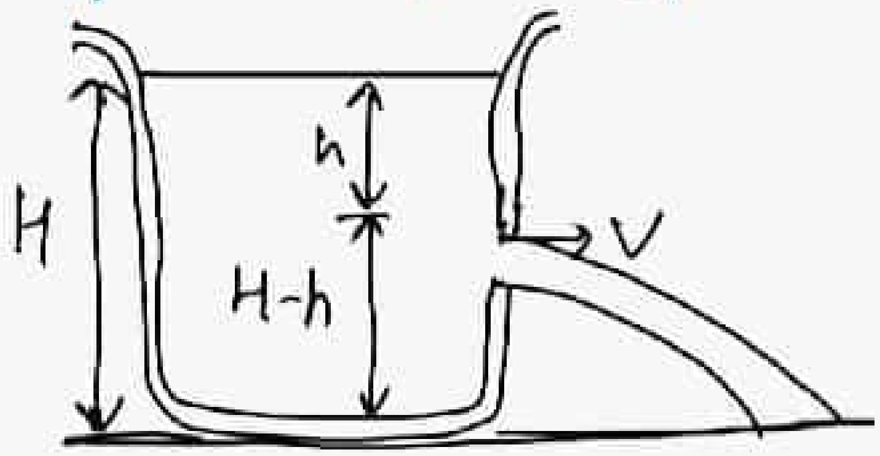
from the continuity eq^n $\left(v_2 = \dfrac{A_1 v_1}{A_2}\right)$

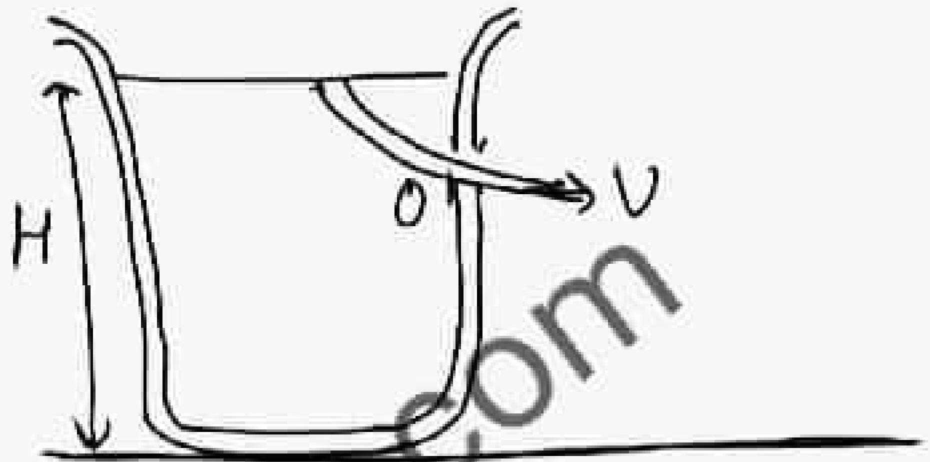
$$\left(P_1 - P_2 = \tfrac{1}{2}\rho v_1^2 \left(\frac{A_1^2}{A_2^2} - 1\right)\right) \quad \text{——} \quad (1)$$

$$v_1 = \sqrt{\frac{2gh}{\left(\frac{A_1}{A_2}\right)^2 - 1}}$$

$$\left(\frac{dV}{dt} = A_1 v_1 = A_1 \sqrt{\frac{2gh}{\left(\frac{A_1}{A_2}\right)^2 - 1}} \right)$$

⇒ Speed of Efflux:-





$$\frac{1}{2}\rho v^2 + \rho g h + p = \text{Constant}$$

$$\rho g h + P_0 = \frac{1}{2}\rho v^2 + P_0 \quad \text{or} \quad v = \sqrt{2gh}$$

⇒ Range [R]:-

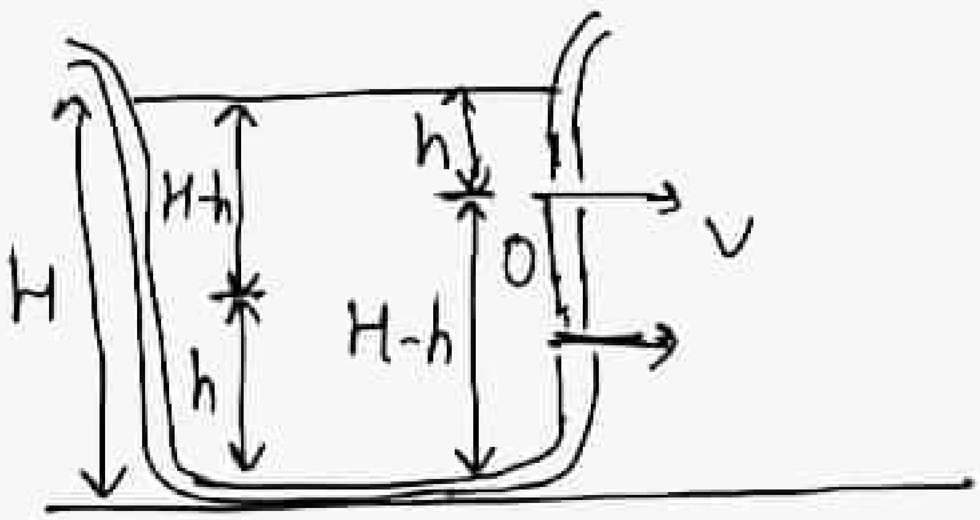
$$(H-h) = \frac{1}{2} g t^2 \quad \text{or} \quad \boxed{t = \sqrt{\frac{2(H-h)}{g}}}$$

Now, considering the horizontal motion,

$$R = vt$$

$$\text{or} \quad R = \left(\sqrt{2gh}\right)\left(\sqrt{\frac{2(H-h)}{g}}\right)$$

or $$R = 2\sqrt{h(H-h)}$$



(i) $R_h = R_{H-h}$ as $R_h = 2\sqrt{h(H-h)}$ and $R_{H-h} = 2\sqrt{(H-h)h}$

(ii) R is maximum at $h = \frac{H}{2}$ and $R_{max} = H$.

Proof $$R^2 = 4(Hh - h^2)$$

for R to be maximum, $\frac{dR^2}{dh} = 0$ or $H - 2h = 0$ or $\left(h = \frac{H}{2}\right)$

That is, R is maximum at $h = \frac{H}{2}$

and $$R_{max} = 2\sqrt{\frac{H}{2}\left(H - \frac{H}{2}\right)} = H$$

Hence Proved

⇒ Time Taken to Empty a Tank:-

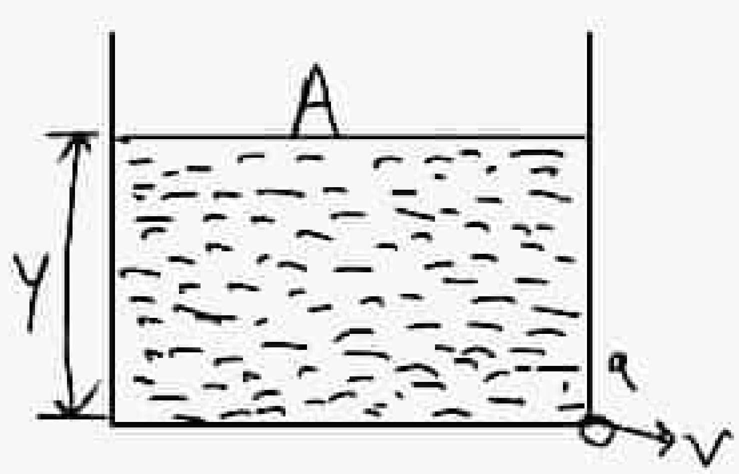
Consider a tank filled with a liquid of density ρ upto a height H. A small hole of area of cross section a is made at the bottom of the tank. The area of cross-section of the tank is A.

Let, at some instant of time the level of liquid in the tank is y. Velocity of efflux at this instant of time would be,

$$v = \sqrt{2gy}$$

Now, at this instant volume of liquid coming out of the hole per second is $\left(\frac{dV_1}{dt}\right)$

Volume of liquid coming down in the tank per second is $\left(\frac{dV_2}{dt}\right)$



$$\frac{dV_1}{dt} = \frac{dV_2}{dt}$$

$$\therefore \quad av = A\left(-\frac{dy}{dt}\right)$$

$$\therefore \quad a\sqrt{2gy} = A\left(-\frac{dy}{dt}\right)$$

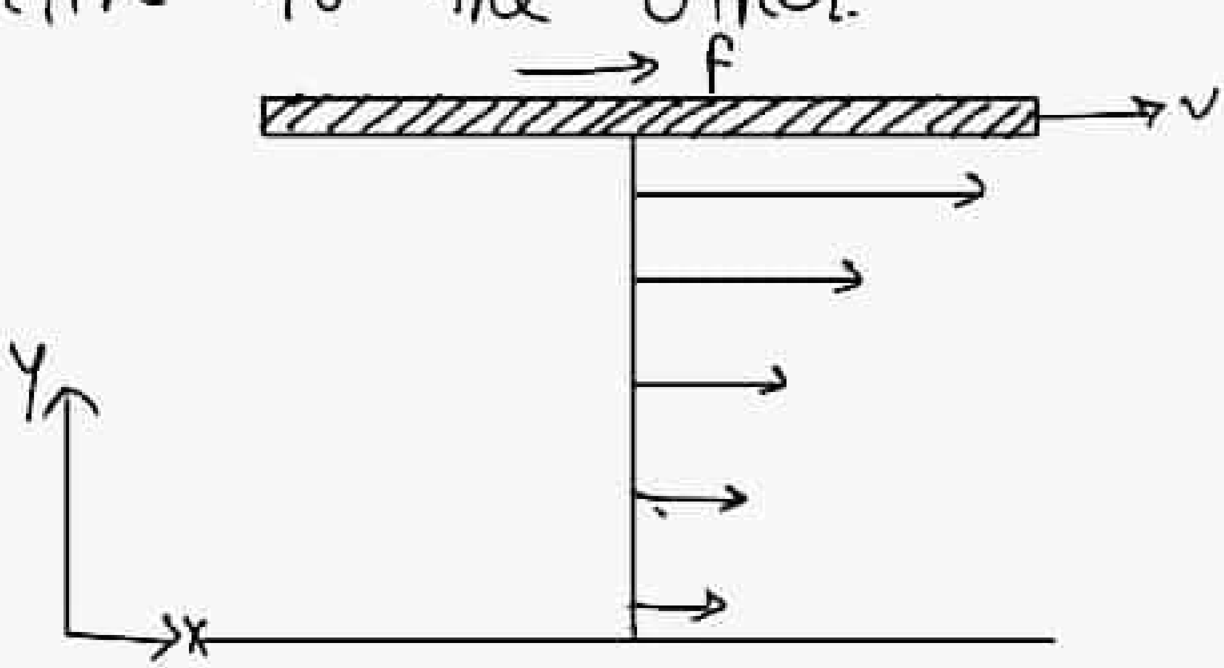
or $$\int_0^t dt = -\frac{A}{a\sqrt{2g}}\int_H^0 y^{-1/2}\,dy \qquad \text{—(1)}$$

$$\therefore \quad t = \frac{2A}{a\sqrt{2g}}\left[\sqrt{y}\right]_0^H$$

$$\therefore \quad \boxed{t = \frac{A}{a}\sqrt{\frac{2H}{g}}}$$

Viscosity :-

Viscosity ~~is~~ internal friction in a fluid. Viscous forces oppose the motion of one portion of a fluid relative to the other.



The simplest example of viscous flow is motion of a fluid between two parallel plates.
The bottom plate is stationary and the top plate moves with constant velocity v. The fluid in contact with each surface has same velocity at that surface. The flow speeds of intermediate layers of fluid increase uniformly from bottom to top, as shown by arrows. So, the fluid layers slide smoothly over one another.

According to Newton, the frictional force f (or viscous force) between two layers depends upon the following factors,

(i) Force f is directly proportional to the area (A) of the layers in contact, i.e.,

$$F \propto A$$

(ii) Force f is directly proportional to the velocity gradient $\left(\frac{dv}{dy}\right)$ between the layers. Combining these two, we have

$$F \propto A \frac{dv}{dy} \quad \text{or} \quad \boxed{f = -\eta A \frac{dv}{dy}}$$

Here, η is constant of proportionality and is called coefficient of viscosity. Its value depends on the nature of the fluid. The negative sign in the above eq^n shows that the direction of viscous force f is opposite to the direction of relative velocity of the layer

The SI unit of η is $N\text{-}s/m^2$. It is also called decapoise or pascal second. Thus,

$$1 \text{ decapoise} = 1\ N\text{-}s/m^2 = 1\ Pa\text{-}s = 10 \text{ poise}$$

The SI unit of viscosity is sometimes referred to as the poiseuille (symbol PI).

Thus, $1\ PI = 1\ N\text{-}s/m^2$

Dimensions of η are $[ML^{-1}T^{-1}]$

Coefficient of viscosity of water at 10°C is $\eta = 13 \times 10^{-3}\ N\text{-}s/m^2$. Experiments show that coefficient of viscosity of a liquid decreases as its temperature rises.

Stoke's law and Terminal Velocity:-

When an object moves through a fluid, it experiences a viscous force which acts in opposite direction of its velocity. The mathematics of the viscous force for an irregular object is difficult, we will consider here only the case of a small sphere moving through a fluid.

The formula for the viscous force on a sphere was first derived by the English physicist G. Stokes in 1843. According to him, a spherical object of radius r moving at velocity v experiences a viscous force given by

$$\boxed{F = 6\pi\eta r v} \quad (\eta = \text{coefficient of viscosity})$$

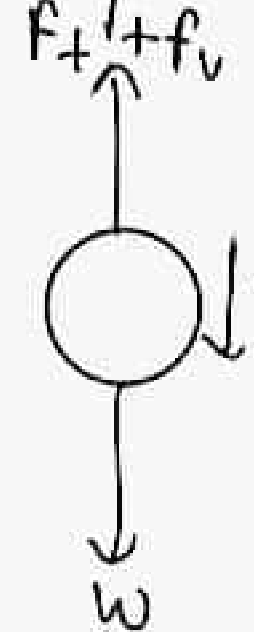
This law is called Stoke's law.

⇒ Terminal Velocity (v_T):-

Consider a small sphere falling from rest through a large column of viscous fluid. The forces acting on the sphere are,

(1) Wgt w of the sphere acting vertically downwards
(2) Upthrust f_t acting vertically upwards
(3) Viscous force f_v acting vertically upwards, i.e. in a direction opposite to velocity of the sphere.

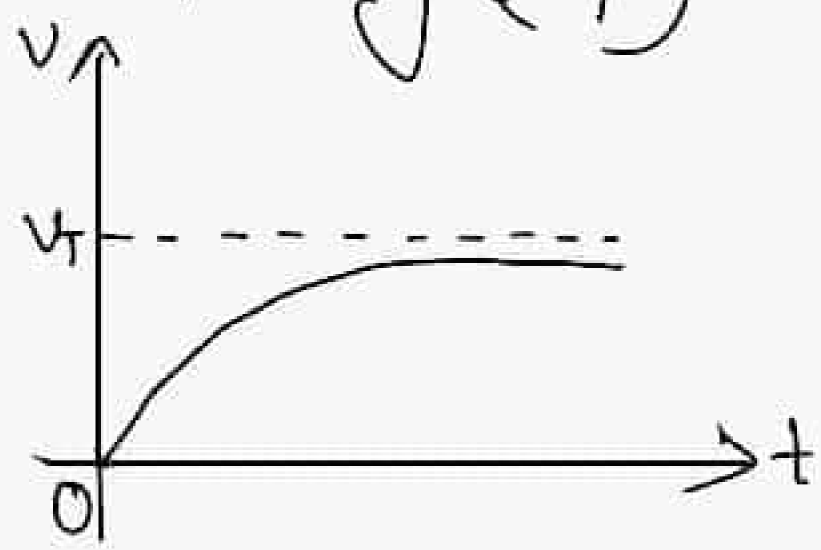
Initially, $f_v = 0$ (as $v = 0$)
and $w > f_t$



and the sphere accelerates downwards. As the velocity of the sphere increases, f_v increases. Eventually a stage is reached when

$$w = f_t + f_v \quad \text{——— (1)}$$

After this net force on the sphere is zero and it moves downwards with a constant velocity called Terminal velocity (v_T)



Substituting proper values in Eq^n (i) we have,

$$\frac{4}{3}\pi r^3 \rho g = \frac{4}{3}\pi r^3 \sigma g + 6\pi\eta r v_T$$

Here, ρ = density of sphere, σ = density of fluid and η = Coefficient of viscosity of fluid

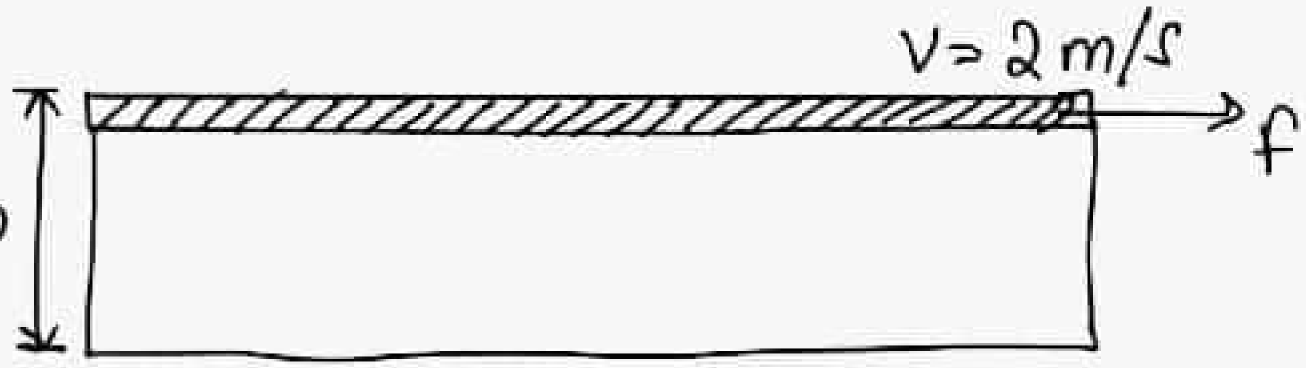
from Eq^n (ii), we get

$$\boxed{v_T = \frac{2}{9}\frac{r^2(\rho-\sigma)g}{\eta}}$$

fig. shows the variation of the velocity v of the sphere with time.

Eg:- A plate of area $2m^2$ is made to move horizontally with a speed of 2m/s by applying a horizontal tangential force over the free surface of a liquid. If the depth of the liquid is 1m and the liquid in contact with the bed is stationary. Coefficient of viscosity of liquid is 0.01 poise. Find the tangential force needed to move the plate.

Sol^n → Velocity gradient $= \frac{\Delta v}{\Delta y} = \frac{2-0}{1-0} = 2 \frac{m/s}{m}$



from, Newton's law of viscous force,

$$|f| = \eta A \frac{\Delta v}{\Delta y}$$

$$= (0.01 \times 10^{-1})(2)(2)$$

$$= 4 \times 10^{-3}\, N$$

So, to keep the plate moving, a force of 4×10^{-3} N must be applied.

Eg:- Two spherical raindrops of equal size are falling vertically through air with a terminal velocity of 1m/s. What would be the terminal speed if these two drops were to coalesce to form a large spherical drop?

Solⁿ → $v_T \propto r^2$ —— (1)

Let r be the radius of small rain drops and R the radius of Large drop.

Equating the Volumes, we have

$$\frac{4}{3}\pi R^3 = 2\left(\frac{4}{3}\pi r^3\right)$$

$$\therefore \quad R = (2)^{1/3} \cdot r \quad \text{or} \quad \frac{R}{r} = (2)^{1/3}$$

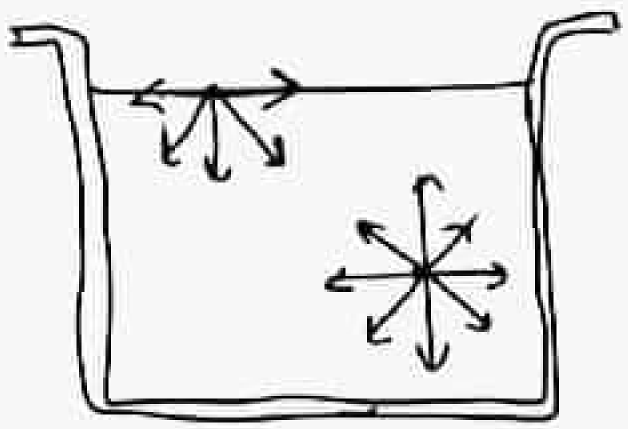
$$\therefore \quad \frac{v_T'}{v_T} = \left(\frac{R}{r}\right)^2 = (2)^{2/3}$$

$$\therefore \quad v_T' = (2)^{2/3} v_T = (2)^{2/3}(1.0) \text{ m/s}$$

$$= 1.587 \text{ m/s} \Rightarrow \text{(Ans)}$$

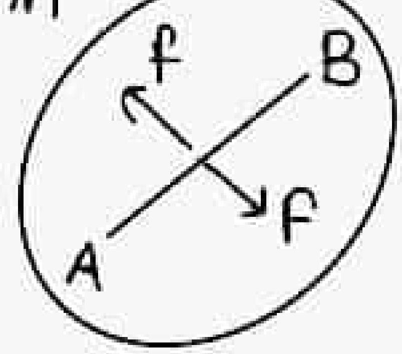
Surface Tension :-

The surface of a liquid behaves like a membrane under tension. The molecules of the liquid exert attractive forces on each other. There is zero net force on a molecule inside the volume of the liquid. But a surface molecule has a net force towards inside of the liquid. Thus, the liquid tends to minimize its surface area, just as a stretched membrane does.



Freely falling raindrops are spherical because a sphere has a smaller surface area for a given volume than any other shape. Hence, the surface tension can be defined as the property of a liquid at rest by virtue of which its free surface behave like a stretched membrane under tension and tries to occupy as small area as possible.

Let an imaginary line AB be dracon in any direction in a liquid surface. The surface on either side of this line exerts a pulling force on the surface on the other side. This force is at right angles to the line AB. The magnitude of this force per unit



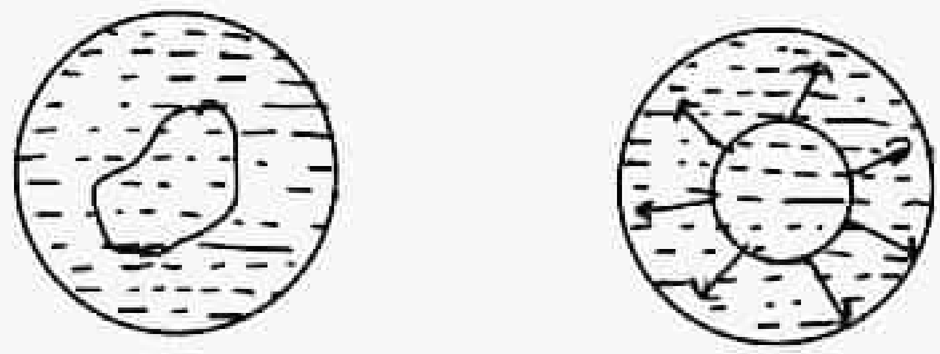
length of AB is taken as a measure of the surface tension of the liquid. Thus, if F be the total force acting on either side of the line AB of length L, then the surface tension is given by,

$$T = \frac{F}{L}$$

Hence, the surface tension of a liquid is defined as the force per unit length in the plane of the liquid surface, acting at right angles on either side of an imaginary line drawn on that surface.

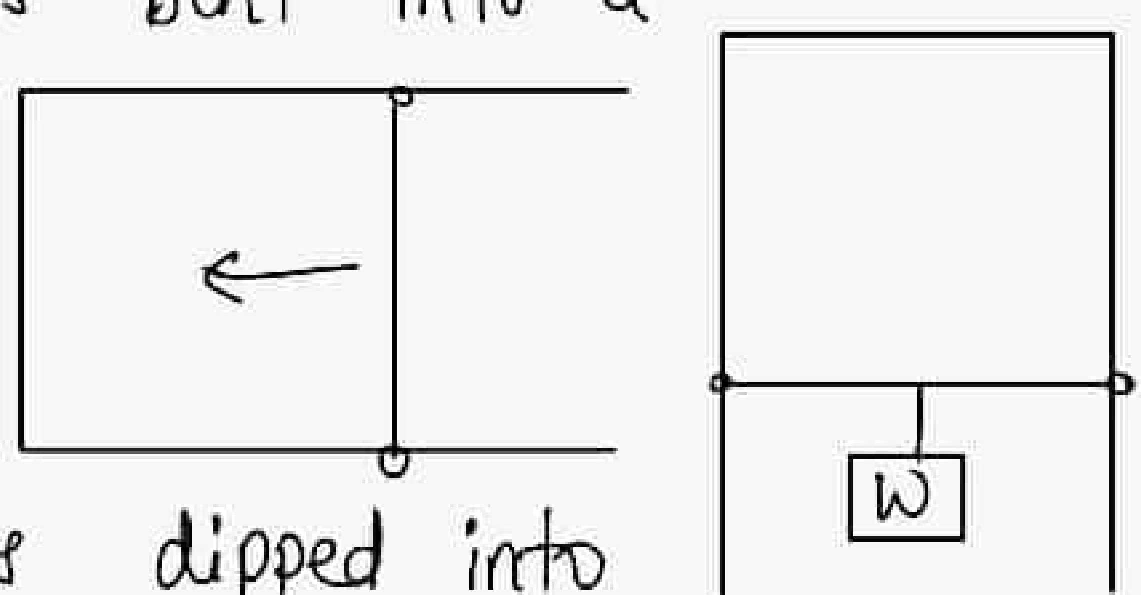
Examples of Surface Tension :-

Eg:- 1 Take a ring of wire and dip it in a soap solⁿ. When the ring is taken out, a soap film is formed. Place a loop of thread gently on the soap film. Now, prick a hole inside the loop. The thread is radially pulled by the film surface outside and it takes a circular shape.



Reason:- Before the pricking, there were surfaces both inside and outside the thread loop. Surfaces on both sides pull it equally and the net force is zero. Once the surface inside was punctured, the outside surface pulled the thread to take the circular shape so that area outside the loop becomes minimum (because for given perimeter area of circle is maximum).

Eg-2. A piece of wire is bent into a U-shaped and a second piece of wire slides on the arms of the U.



When the apparatus is dipped into a soap solⁿ and removed, a liquid film is formed. The film exerts a surface tension force on the slider and if the frame is kept in a horizontal position, the slider quickly slides towards the closing arm of the frame. If the frame is kept vertical, one can have some weight to keep it in equilibrium. This shows that the soap surface in contact with the slider pulls it parallel to the surface.

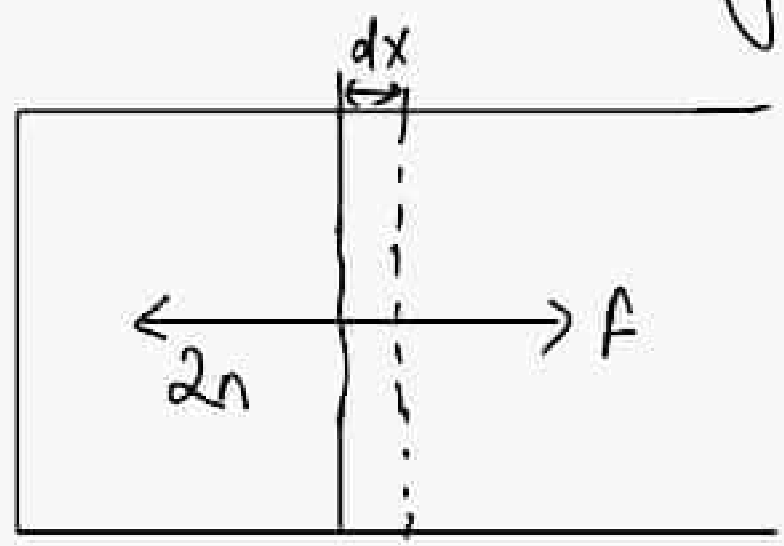
Eg-3. Small mercury droplets are spherical and Larger ones tend to flatten.

Reason:- Small mercury droplets are spherical because the forces of surface tension tend to reduce their area to a minimum value and a sphere has minimum surface area for a given volume.

Larger drops of mercury are flattened due to the large gravitational force acting on them. Here the shape is such that the sum of the gravitational potential energy and the surface potential energy must be minimum. Hence the centre of gravity moves down as low as possible. This explains flattening of the larger drops.

Surface Energy :-

When the surface area of a liquid is increased, the molecules from the interior rise to the surface. This requires work against force of attraction of the molecules just below the surface. This work is stored in the form of potential energy. Thus, the molecules in the surface have some additional energy due to their position. This additional energy per unit area of the surface is called "Surface Energy". The surface energy is related to the surface tension as discussed below:-



let a liquid film be formed on a wire frame and a straight wire of length l can slide on this wire frame as shown in fig. The film has two surfaces and both the surfaces are in contact with the sliding wire and hence, exert forces of surface tension on it. If T be the surface tension of the solⁿ, each surface will pull the wire parallel to itself with a force Tl. Thus, net force on the wire due to both the surfaces is $2Tl$. One has to apply an external force F equal and opposite to it to keep the wire in equilibrium. Thus,

$$F = 2Tl$$

Now, Suppose the wire is move through a small distance dx, the work done by the force is,

$$dW = F\, dx = (2Tl)\, dx$$

But $(2l)(dx)$ is the total increase in area of both the surfaces of the film. let it be dA. Then,

$$dW = T\, dA$$

or, $$\boxed{T = \frac{dW}{dA} \text{ or } \frac{\Delta W}{\Delta A}}$$

Thus, the surface tension T can also be defined as the work done in increasing the surface area by Unity. Further, since there is no change in kinetic energy, the work done by the external force is stored as the potential energy of the new surface

$$\therefore \quad \boxed{T = \frac{dU}{dA} \text{ or } \frac{\Delta U}{\Delta A}} \qquad (\text{as } dW = dU)$$

Thus, the surface tension of a liquid is equal to the surface energy per unit surface area.

Excess Pressure Inside a Bubble or liquid Drop:-

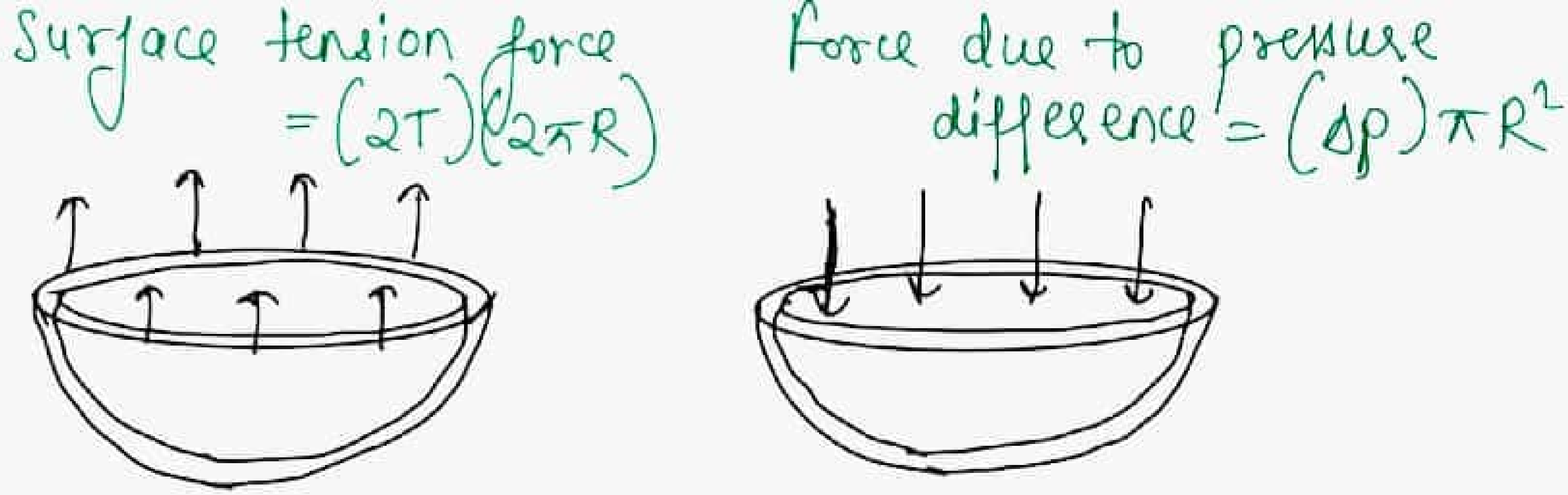
Surface tension causes a pressure difference between the inside and outside of a soap bubble or a liquid drop.

⟹ Excess Pressure Inside Soap Bubble:-

A soap bubble consists of two spherical surface films with a thin layer of liquid between them. Because of surface tension, the film tend to contract in an attempt to minimize their surface area. But as the bubble contracts, it compresses the inside air, eventually increasing the interior pressure to a level that prevents further contraction.

We can derive an expression for the excess pressure inside a bubble in terms of its radius R and the surface tension T of the liquid.

Each half of the soap bubble is in equilibrium. The forces at the flat circular surface where this half joins the upper half are-



(a) The upward force of surface tension. The total surface tension force for each surface (inner and outer) is $T(2\pi R)$, for a total of $(2T)(2\pi R)$.

(b) downward force due to pressure difference. The magnitude of this force is $(\Delta p)(\pi R^2)$. In equilibrium these two forces have equal magnitude.

$$\therefore \quad (2T)(2\pi R) = (\Delta p)(\pi R^2)$$

or
$$\boxed{\Delta p = \frac{4T}{R}}$$

Note:- Suppose, the pressure inside the air bubbles is p, then

$$p - p_0 = \frac{4T}{R}$$

⇒ Excess Pressure Inside a liquid Drop:-

A liquid drop has only one surface film. Hence, the surface tension force is $T(2\pi R)$, half that for a soap bubble. Thus, in equilibrium,

$$T(2\pi R) = \Delta p(\pi R^2)$$

or

$$\boxed{\Delta p = \frac{2T}{R}}$$

Eg:- How much work will be done in increasing the diameter of a soap bubble from 2 cm to 5 cm? Surface tension of soap solⁿ is 3.0×10^{-2} N/m.

Solⁿ→ Soap bubble has two surfaces. Hence,

$$W = T\Delta A \qquad \left(T = \frac{\Delta W}{\Delta A}\right)$$

Here, $\Delta A = 2[4\pi\{(2.5\times10^{-2})^2 - (1.0\times10^{-2})^2\}]$

$= 1.32 \times 10^{-2}\ m^2$

$\therefore\quad W = (3.0\times10^{-2})(1.32\times10^{-2})$ J

$= 3.96 \times 10^{-4}$ J

Eg:- What should be the pressure inside a small air bubble of 0.1 mm radius situated just below the water surface. Surface tension of water $= 7.2 \times 10^{-2}$ N/m and atmospheric pressure $= 1.013 \times 10^{5}$ N/m^2

Sol^n → Surface tension of water $T = 7.2 \times 10^{-2}$ N/m

Radius of air bubble $R = 0.1\,mm = 10^{-4}\,m$

The excess pressure inside the air bubble is given by,

$$P_2 - P_1 = \frac{2T}{R}$$

∴ Pressure inside the air bubble,

$$P_2 = P_1 + \frac{2T}{R}$$

Substituting the values, we have,

$$P_2 = (1.013 \times 10^5) + \frac{(2 \times 7.2 \times 10^{-2})}{10^{-4}}$$

$$= 1.027 \times 10^5 \ N/m^2$$

Capillary Rise or fall:-

⇒ Cohesive and Adhesive forces:-

The force of attraction between the molecules of the same substance is called "Cohesion".
In case of solids, the force of Cohesion is very large and due to this solids have definite shape and size. On the other hand, the force of Cohesion in case of liquids is weaker than that of solids. Hence, liquids do not have definite shape but have definite volume. The force of Cohesion is negligible in case of gases. Because of this fact, gases have neither fixed shape nor volume.

⇒ Contact Angle :-

When a liquid surface touches a solid surface, the shape of the liquid surface near the contact is generally curved. When a glass plate is immersed in water, the surface near the plate becomes concave. On the other hand, if glass plate is immersed in mercury, the surface is depressed near the plate.

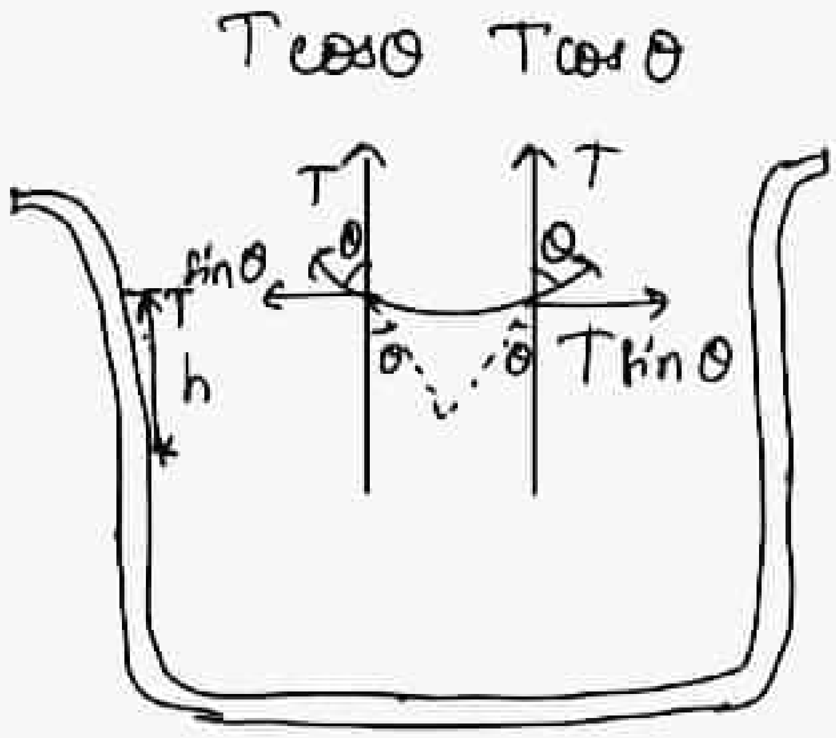
The angle between the tangent planes at the solid surface and the liquid at the contact is called the contact angle. In this case the tangent plane to the solid surface is to be dracon towards the liquid and the tangent plane to the liquid is to be drawn away from the solid.

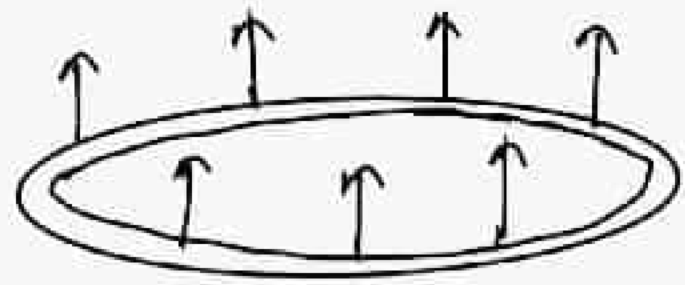
$\theta < 90°$ $\theta > 90°$

Those liquids which wet the walls of the container (say in case of water and glass) have meniscus concave upwards and their values of angle of contact is less than 90° (also called acute angle). However, those liquids which don't wet the walls of the container (say in case of mercury and glass) have meniscus convex upwards and their value of angle of contact is greater than 90° (also called obtuse angle). The angle of contact of mercury with glass is about 140°, whereas the angle of contact of water with glass is about 8°. But, for pure water, the angle of contact θ with glass is taken as 0°.

Proof for the formula of Capillary Rise:-

When the contact angle is less than 90°, the total surface tension force just balances the extra weight of the liquid in tube. Due to the surface tension of water, a force equal to T per unit length acts at all points of the circle. If the angle of contact is θ, then this force is directed inward at an angle θ from the wall of the tube.





Eg:- A capillary tube whose inside radius is 0.5 mm is dipped in water having surface tension 7.0×10^{-2} N/m. To what height is the water raised above the normal water level? Angle of contact of water with glass is 0°. Density of water is 10^3 kg/m^3 and $g = 9.8$ m/s^2.

Sol^n:- $h = \dfrac{2T\cos\theta}{r\rho g}$

Substituting the proper values, we have

$$h = \frac{(2)(7.0 \times 10^{-2})\cos 0^\circ}{(0.5 \times 10^{-3})(10^3)(9.8)}$$

$$= 2.86 \times 10^{-2}\ \text{m} = 2.86\ \text{cm} \Rightarrow \text{(Ans)}$$