OSCILLATIONS

- Periodic motion – Any motion that repeats itself over and over again at regular intervals of time is called Periodic or harmonic motion. The smallest intervel of time after which the motion is repeated is called its time Period. The time Period is denoted by T and its S.I. unit is second.

- Oscillatory motion – If a body moves back and forth repeatedly about its mean Position. Its motion is said to be oscillatory or vibratory or harmonic motion.

- Distinction between Periodic and oscillatory motion – Every oscillatory motion is necessarily Periodic because it is repeated at regular intervals of time. In addition it is bounded about one mean Position. But every Periodic motion need not be oscillatory.

- Periodic function – Any function that repeats itself at regular intervals of its argument is called a Periodic function.

$$F(t) = \sin\frac{2\pi t}{T} \quad \text{and} \quad g(t) = \cos\frac{2\pi t}{T}.$$

The sine and cosine functions are Periodic with Period T. The Periodic function which can be represented by a sine or cosine curve are called harmonic function. All harmonic functions are necessarily Periodic but all Periodic functions are not harmonic.
The Periodic functions which cannot be represented by single sine or cosine functions are called non-harmonic functions.

- Fourier theorem – This theorem states that any arbitary function $F(t)$ with Period T can be expressed as the unique Combination of sine and Cosine function $f_n(t)$ and $g_n(t)$ with Suitable Coefficients.

It can be expressed as

$$F(t) = b_0 + b_1 \cos\frac{2\pi t}{T} + b_2 \cos\frac{4\pi t}{T} + b_3 \cos\frac{6\pi t}{T} + \cdots$$

$$\cdots + a_1 \sin\frac{2\pi t}{T} + a_2 \sin\frac{4\pi t}{T} + a_3 \sin\frac{6\pi t}{T} + \cdots$$

$\because \omega = \frac{2\pi}{T}$ then:

$$F(t) = b_0 + b_1 \cos\omega t + b_2 \cos 2\omega t + b_3 \cos 3\omega t + \cdots$$

$$\cdots + a_1 \sin\omega t + a_2 \sin 2\omega t + a_3 \sin 3\omega t + \cdots$$

$$= b_0 + \sum_{n=1}^{n} b_n \cos n\omega t + \sum_{n=1}^{n} a_n \sin n\omega t$$

The Coefficient $a_1, a_2, a_3 \cdots, b_1, b_2, b_3 \cdots$ are called fourier Coefficient. These Coefficient can be determined uniquely by a mathematical method called Fourier analysis.

Let all the fourier Coefficients except a_1, b_1 are zero than:

$$F(t) = a_1 \sin\frac{2\pi t}{T} + b_1 \cos\frac{2\pi t}{T}$$

This equation is a special Periodic motion Called Simple harmonic motion (S.H.M.)

— Simple Harmonic motion — A Particle is said to execute simple harmonic motion if it moves to and fro about a mean Position under the action of a restoring force which is directly Proportional to its displacement from the mean Position and is always directed towards the mean Position.

Restoring force $\propto$ displacement.

$$F \propto x. \quad \text{or} \quad F = -kx.$$

This equation define S.H.M.. Here k is a Positive Constant Called force Constant or spring factor. The S.I. unit of k is.

N/m. The negative sign of the equation shows that the restoring force F always acts in the opposite direction of the displacement x.

Now according to Newton's second law of motion.

$$F = ma.$$

$$\therefore \quad ma = -kx.$$

$$a = -\frac{k}{m}.x. \qquad \therefore \quad a \propto x.$$

Importent features.

1. The motion of the particle is periodic.
2. It is the oscillatory motion of simplest kind in which the particle oscillate back and forth about its mean position with constant amplitude and frequency.
3. Restoring force acting on the particle is proportional to its displacement from the mean position.
4. The force acting on the particle always oppose the increase in its displacement.
5. A simple harmonic motion can always be expressed in term of a single harmonic function of sine or cosine.

— Differential equation of S.H.M.

In S.H.M., the restoring force acting on the particle is proportional to its displacement, thus.

$$F = -kx.$$

By Newton's second law

$$F = ma = m.\frac{d^2x}{dt^2}$$

then. $m.\frac{d^2x}{dt^2} = -kx.$ or. $\frac{d^2x}{dt^2} = -\frac{k}{m}.x.$

But $\frac{k}{m} = \omega^2$ then. $\frac{d^2x}{dt^2} = -\omega^2 x.$

$$\frac{d^2x}{dt^2} + \omega^2 x = 0 \qquad — (1)$$

This is the differential equation of S.H.M. Here ω is the angular frequency. Clearly, x should be such a function whose second derivative is equal to the function itself multiplied with a negative constant. So a possible solution of eq. ① may be

$$x = A\cos(\omega t + \phi_0)$$

then $\frac{dx}{dt} = -\omega A \sin(\omega t + \phi_0)$

and $\frac{d^2x}{dt^2} = -\omega^2 A \cos(\omega t + \phi_0) = -\omega^2 x.$

or. $\frac{d^2x}{dt^2} + \omega^2 x = 0$

Hence the solution of eq. ① is $x = A\cos(\omega t + \phi_0)$ — ②

Here A = Amplitude of the oscillating particle

$\phi = \omega t + \phi_0$ is the phase of the oscillating particle

ϕ_0 is the initial (at $t=0$) phase.

Time Period of S.H.M — If we replace t by $t + \frac{2\pi}{\omega}$ in eq. ②, we get

$$x = A\cos\left[\omega\left(t + \frac{2\pi}{\omega}\right) + \phi_0\right]$$
$$= A\cos(\omega t + 2\pi + \phi_0)$$
$$= A\cos(\omega t + \phi_0)$$

So the motion repeats after time interval $\frac{2\pi}{\omega}$. Hence $\frac{2\pi}{\omega}$ is the time period of S.H.M.

$$T = \frac{2\pi}{\omega} = \frac{2\pi}{\sqrt{K/m}} \qquad \because \omega^2 = \frac{K}{m}.$$

$$= 2\pi\sqrt{\frac{m}{K}} = 2\pi\sqrt{\frac{\text{Inertia factor}}{\text{Spring factor}}}.$$

Q. Which of the following functions of time represent (a) periodic and (b) non-periodic motion. Give the period for each case of periodic motion [ω is any positive constant]

(I) $\sin\omega t + \cos\omega t$ (II) $\sin\omega t + \cos 2\omega t + \sin 4\omega t$.

(III) $e^{-\omega t}$ (IV) $\log(\omega t)$

Ans. (i) Here $x(t) = \sin\omega t + \cos\omega t$

multiply and divide by $\sqrt{2}$.

$$= \sqrt{2}\left[\frac{1}{\sqrt{2}}\sin\omega t + \frac{1}{\sqrt{2}}\cos\omega t\right]$$

$$= \sqrt{2}\left[\sin\omega t \cos\frac{\pi}{4} + \sin\frac{\pi}{4}\cos\omega t\right]$$

$$= \sqrt{2}\sin\left(\frac{\pi}{4} + \omega t\right)$$

Moreover.

$$x\left(t + \frac{2\pi}{\omega}\right) = \sqrt{2}\sin\left(\frac{\pi}{4} + \omega\left(t + \frac{2\pi}{\omega}\right)\right)$$
$$= \sqrt{2}\sin\left(\frac{\pi}{4} + \omega t + 2\pi\right)$$
$$= \sqrt{2}\sin\left(\pi/4 + \omega t\right)$$
$$= x(t)$$

Hence $\sin\omega t + \cos\omega t$ is a Periodic function with time Period equal to $\frac{2\pi}{\omega}$.

(ii) Here $x(t) = \sin\omega t + \cos 2\omega t + \sin 4\omega t$.

$\sin\omega t$ is a Periodic function with Period.

$$T = \frac{2\pi}{\omega}.$$

$\cos 2\omega t$ is a Periodic function with Period.

$$= \frac{2\pi}{2\omega} = \frac{\pi}{\omega} = \frac{T}{2}$$

$\sin 4\omega t$ is a Periodic function with Period

$$\frac{2\pi}{4\omega} = \frac{\pi}{2\omega} = \frac{T}{4}$$

Clearly the entire function $x(t)$ repeats after a minimum time $T = \frac{2\pi}{\omega}$. Hence the function is Periodic.

(iii) The function $e^{-\omega t}$ decrease monotonically to zero as $t \to \infty$. It is an exponential function with a negative exponent of e where $e = 2.71828$. It never repeat its value. So it is non Periodic.

(IV) The function $\log(\omega t)$ increases monotonically with time. As $t \to \infty$ $\log(\omega t) \to \infty$, it never repeats its value, so it is non-Periodic.

Q. Which of the following functions of time represent (a) simple harmonic (b) Periodic but not S.H.F and (c) non-Periodic motion. Give Period for each case of Periodic motion [ω is any Positive constant]

(I) $\sin \omega t - \cos \omega t$ (II) $\sin^3 \omega t$

(III) $3\cos(\pi/4 - 2\omega t)$ (IV) $\cos \omega t + \cos 3\omega t + \cos 5\omega t$.

(V) $\exp(-\omega^2 t^2)$ (VI) $1 + \omega t + \omega^2 t^2$.

An. I Here $x(t) = \sin \omega t - \cos \omega t$.

multiply and divide by $\sqrt{2}$

$$= \sqrt{2}\left[\frac{1}{\sqrt{2}} \sin \omega t - \frac{1}{\sqrt{2}} \cos \omega t\right]$$

$$= \sqrt{2}\left[\cos\frac{\pi}{4} \sin \omega t - \sin\frac{\pi}{4} \cos \omega t\right]$$

$$= \sqrt{2}\left[\sin(\omega t - \pi/4)\right]$$

$x\left(t + \frac{2\pi}{\omega}\right)$

then $= \sqrt{2} \sin\left(\omega\left(t + \frac{2\pi}{\omega}\right) - \pi/4\right)$

$$= \sqrt{2} \sin(\omega t + 2\pi - \pi/4)$$

$$= \sqrt{2} \sin(\omega t - \pi/4)$$

$$= x(t)$$

Hence the given function represents a simple harmonic motion with $T = \frac{2\pi}{\omega}$ and Phase angle $= -\frac{\pi}{4}$ or $\frac{7\pi}{4}$

(II) $x(t) = \sin^3 \omega t$.

$$= \frac{1}{4}\left[3 \sin \omega t - \sin 3\omega t\right]$$

It represent two seperate simple harmonic motions but their combination does not represent S.H.M.

Period of $\frac{3}{4} \sin \omega t = \frac{2\pi}{\omega} = T$.

Period of $\frac{1}{4} \sin 3\omega t = \frac{2\pi}{3\omega} = \frac{T}{3}$.

thus the minimum time after which the combine function repeats is $T = \frac{2\pi}{\omega}$. Hence the given function is periodic but not simple harmonic.

(iii) $x(t) = 3\cos(\pi/4 - 2\omega t)$

$$= 3\cos[-(2\omega t - \pi/4)] = 3\cos(2\omega t - \pi/4)$$

It represent S.H.M with time period $\frac{2\pi}{2\omega} = \frac{\pi}{\omega}$

(iv) $x(t) = \cos\omega t + \cos 3\omega t + \cos 5\omega t$.

$\cos\omega t$ represents S.H.M with period $= \frac{2\pi}{\omega} = T$.

$\cos 3\omega t$ represents S.H.M with period $= \frac{2\pi}{3\omega} = \frac{T}{3}$

$\cos 5\omega t$ represent S.H.M. with period $= \frac{2\pi}{5\omega} = \frac{T}{5}$.

The minimum time after which the combination function repeats its value is T. The given function is periodic, not S.H.M.

(v) $x(t) = \exp(-\omega^2 t^2) = e^{-\omega^2 t^2}$

It is an exponential function. It decreases monotonically to zero as $t \to \infty$. It never repeat its value. It is a non-periodic function.

(vi) $x(t) = 1 + \omega t + \omega^2 t^2$

As t increases $x(t)$ increases monotonically. Again as $t \to \infty$, $x(t) \to \infty$. the function never repeats its value. So $(x(t))$ is non-periodic

— Some important terms connected with S.H.M.

1. Harmonic oscillator — A particle executing simple harmonic motion is called harmonic oscillator.

2. Displacement — The distance of the oscillating particles from its mean position at any instant is called its displacement. It is denoted by x.

3. Amplitude — The maximum displacement of the oscillating particle on either side of its mean position is called.

its amplitude. It is denoted by A. Thus $x_{max} = \pm A$.

4. Oscillation or cycle – One complete back and fro motion of a particle starting and ending at the same point is called a cycle or oscillation or vibration.

5. Time Period – The time taken by a particle to complete one oscillation is called its time period. or, It is the smallest time interval after which the oscillatory motion repeats. It is denoted by T.

6. Frequency – It is defined as the number of oscillations completed per unit time by a particle. It is denoted by ν (nu). Frequency is equal to the reciprocal of time period. The unit of frequency is (second^{-1}) or hertz (Hz)

7. Angular frequency – It is the quantity obtained by multiplying frequency (ν) by a factor of 2π. It is denoted by ω. The S.I. unit of angular frequency is (rad/second)

$$\omega = 2\pi\nu = \frac{2\pi}{T}$$

8. Phase – The phase of a vibrating particle at any instant gives the state of the particle as regards its position and the direction of motion at that instant.

Suppose a S.H. equation is represented by

$$x = A\cos(\omega t + \phi_0)$$

then phase of the particle is $\phi = \omega t + \phi_0$

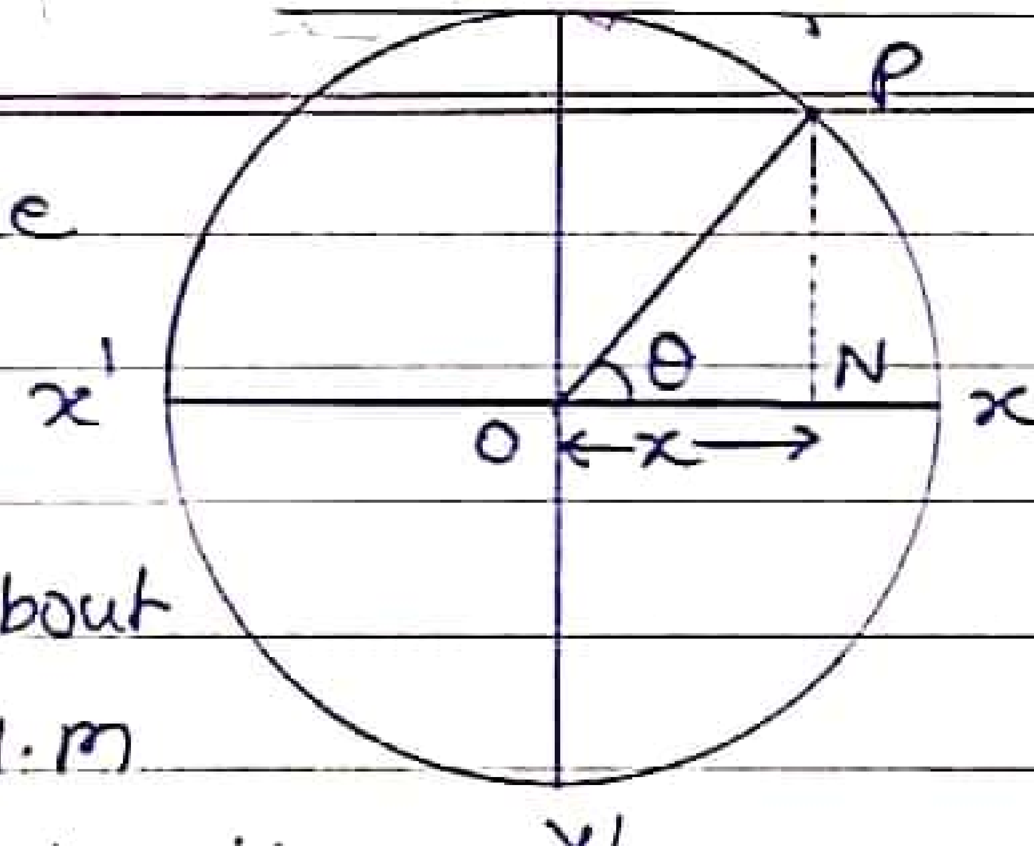
9. Initial phase or epoch – The phase of a vibrating particle corresponding to time $t=0$ is called initial phase or epoch.

At $t=0$ $\quad \phi = \phi_0$

– Uniform circular motion and S.H.M.

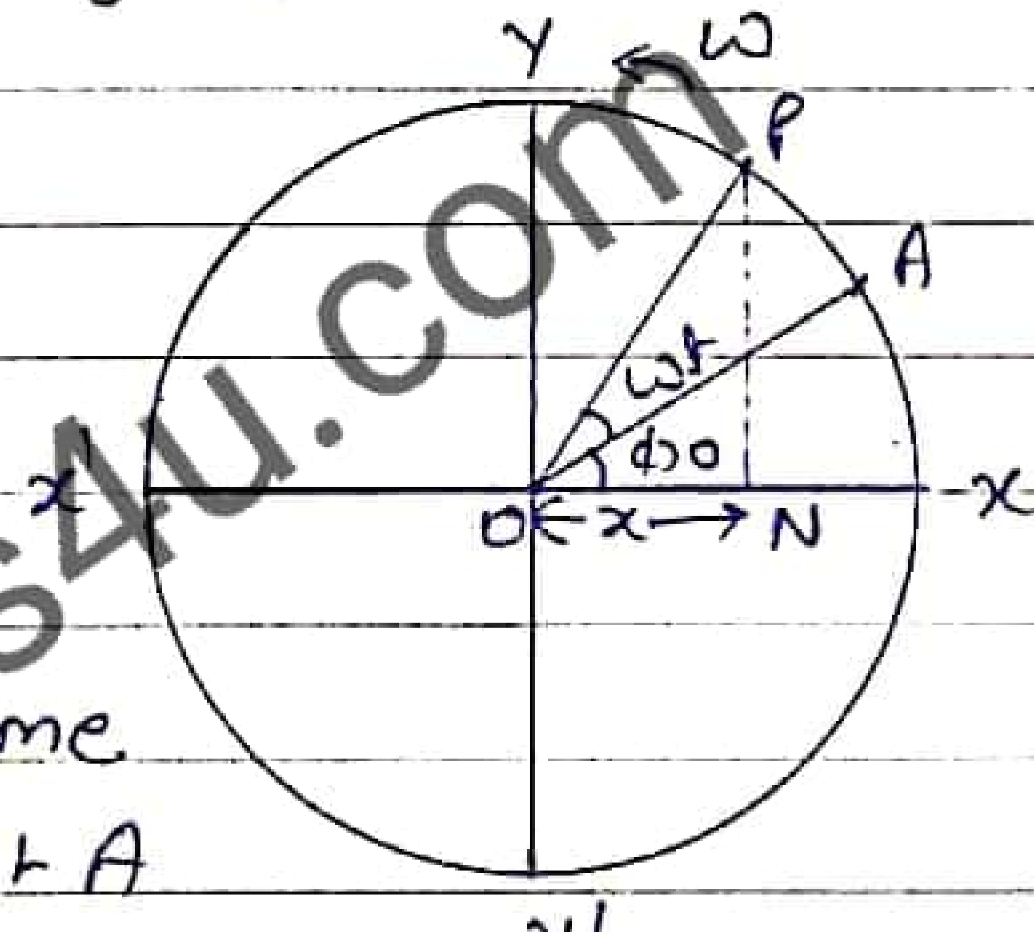
Consider a particle P moving along a circle of radius A with uniform angular velocity ω. Let N be the foot of of the perpendicular drawn from the point P to the diameter xx'. Then N is called the projection of P on the diameter.

As P moves along the circumference of the circle, N moves to and fro about the the Point O along the diameter xx'. The motion of N about O is called to be S.H. Hence S.H.M maybe defined as the Projection of uniform circular motion upon a diameter of a circle. The Particle P is called reference Particle and the circle along which the Particle P revolves is called circle of reference.



Displacement in S.H.M.

Consider a Particle moving in anti clockwise direction with uniform angular velocity ω along a circle of radius A and center O. Suppose at time $t=0$, the reference Particle is at Point A such that $\angle AOx = \phi_0$. At any time t, the Particle reaches the Point P such that $\angle POA = \omega t$.



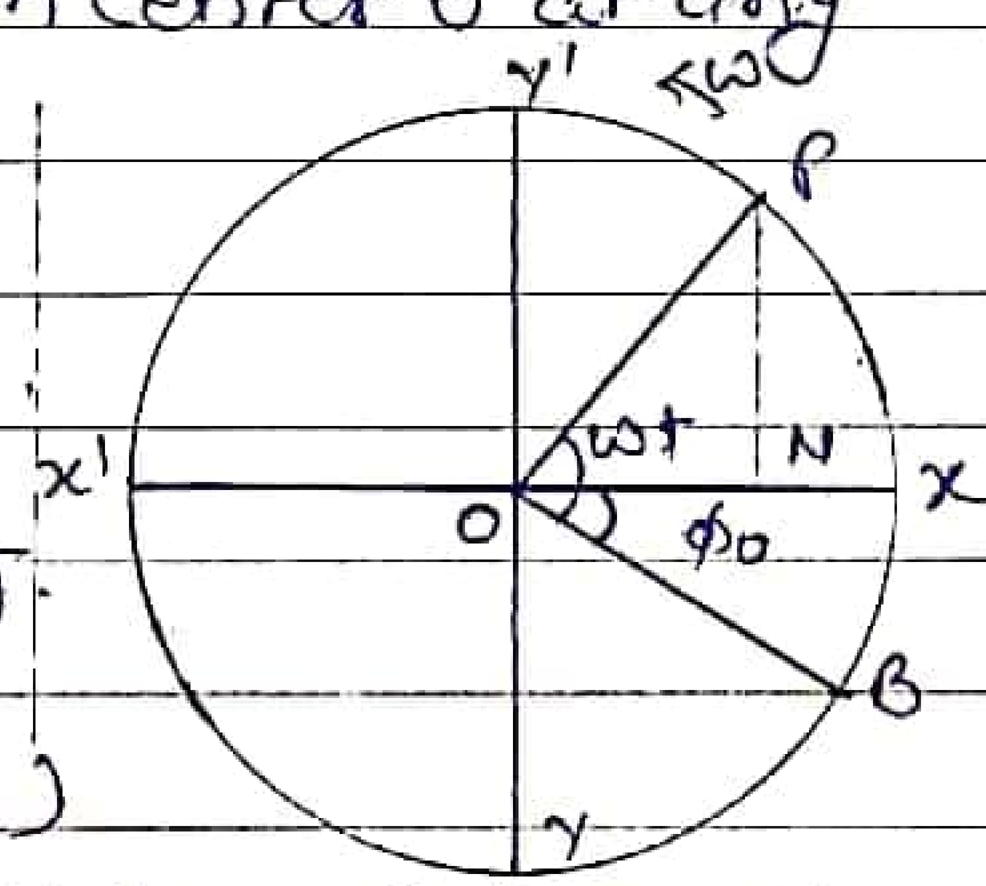
Draw $PN \perp xx'$

The displacement of Projection N from center O at any instant t is $x = ON$

In right tringle ONP.

$$\angle PON = \omega t + \phi_0$$

$$\cos(\omega t + \phi_0) = \frac{ON}{OP} = \frac{x}{A}$$



then $x = A\cos(\omega t + \phi_0)$

This equation gives displacement of a Particle in S.H.M. at any instant t. The Quantity $\omega t + \phi_0$ called Phase and ϕ_0 is called initial Phase. The Quantity A called Amplitude

If the Particle start from Point B, such that $\angle BOx = \phi_0$ and $\angle BOP = \omega t$ then $\angle PON = \omega t - \phi_0$

So $x = A\cos(\omega t - \phi_0)$

Q. Show that a linear Combination of sine and Cosine function like $x(t) = a\sin\omega t + b\cos\omega t$ represent a simple Harmonic motion. Determine its amplitude and Phase Constant

Ans $$x = a\sin\omega t + b\cos\omega t \quad -(1)$$

Differentiate w.r.t. time t

$$\frac{dx}{dt} = \omega a\cos\omega t - \omega b\sin\omega t.$$

Again differentiate w.r.t time t.

$$\frac{d^2x}{dt^2} = -\omega^2 a\sin\omega t - \omega^2 b\cos\omega t.$$

$$= -\omega^2(a\sin\omega t + b\cos\omega t)$$

$$= -\omega^2 x.$$

then. acceleration $\propto$ displacement.

Hence the equation (1) represent S.H.M.

To determine its amplitude and Phase Constant we Put

$$a = A\cos\phi \quad -(2)$$

$$b = A\sin\phi \quad -(3)$$

then $$x = A\cos\phi\sin\omega t + A\sin\phi\cos\omega t.$$

$$= A(\sin\phi\cos\omega t + \cos\phi\sin\omega t)$$

$$= A\sin(\omega t + \phi)$$

This again shows that eq. (1) represents S.H.M. of Amplitude A and Phase Constant ϕ

Squaring and adding eq. (2) and (3)

$$a^2 + b^2 = A^2(\cos^2\phi + \sin^2\phi) = A^2$$

$$\therefore A = \sqrt{a^2 + b^2}$$

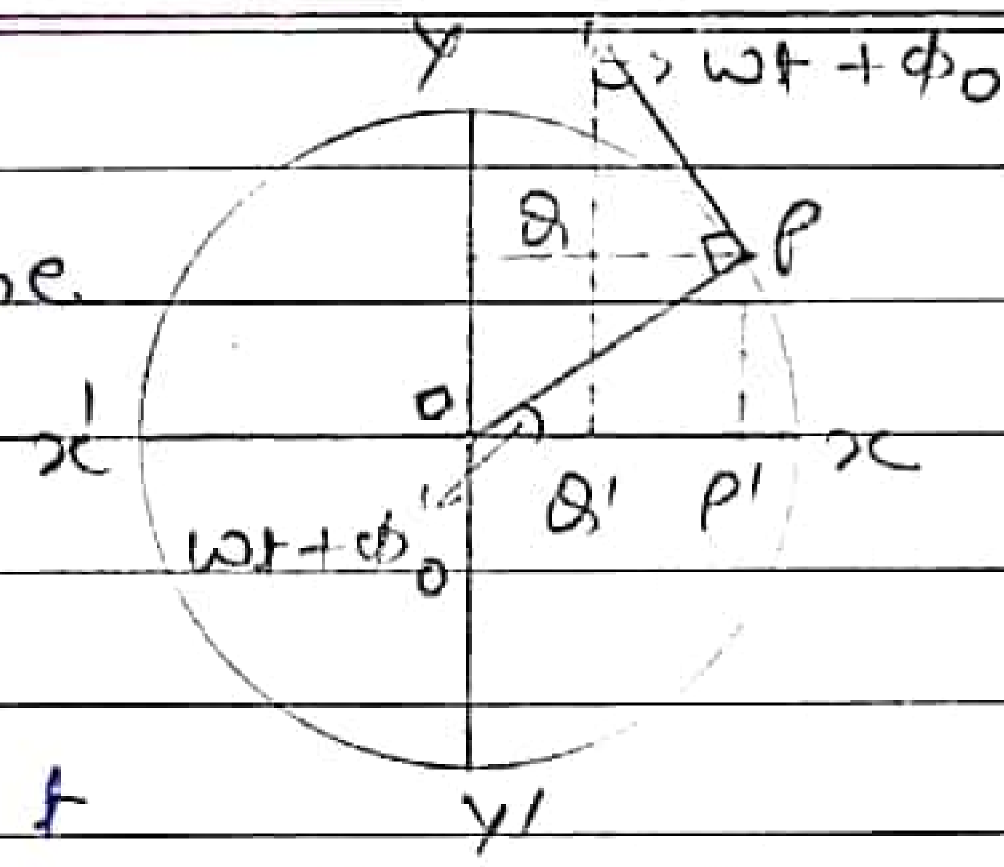
Dividing (3) by (2), we get.

$$\frac{a}{b} = \frac{A\sin\phi}{A\cos\phi} = \tan\phi$$

$$\phi = \tan^{-1}\frac{b}{a}.$$

— Velocity in S.H.M.— Consider a Particle P moving with uniform angular speed ω in a circle of radius A, Its velocity vector $\vec{v}$ is directed along the tangent and the magnitude of this velocity vector is.

$V =$ Angular velocity x radius $= \omega A$.
PP' and QQ' are Perpendiculars on the diameter xx'. The motion of P' is S.H.
Clearly, the instantaneous velocity of a particle executing S.H.M. will be



$V(t) =$ velocity of Particle P' at any instant t
$= $ Projection of the velocity V of the reference Particle P
$= P'Q' = PQ = -V \sin(\omega t + \phi_0)$
$$V(t) = -\omega A \sin(\omega t + \phi_0)$$
the negative shows that the velocity of P' is directed towards left in the negative x-direction.
$$V(t) = -\omega A \sqrt{1 - \cos^2(\omega t + \phi_0)} = -\omega A \sqrt{1 - \frac{x^2}{A^2}}.$$
$$= -\frac{\omega A}{A} \sqrt{A^2 - x^2} = -\omega \sqrt{A^2 - x^2}.$$

Case I

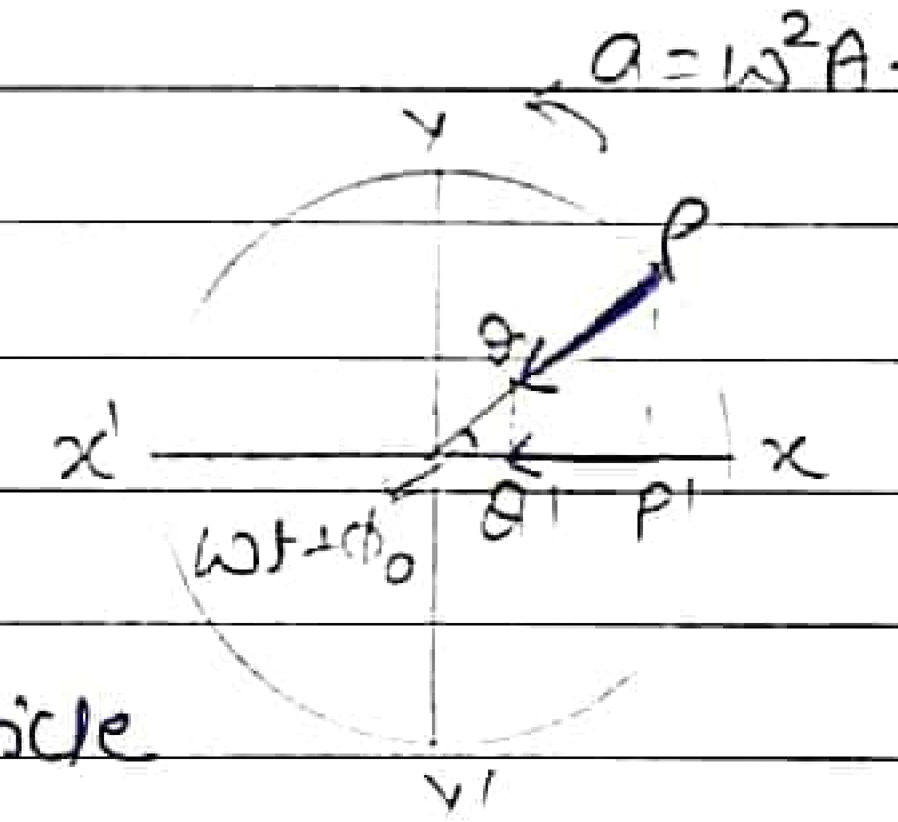
When the Particle is at the mean Position then $x = 0$ so $V(t) = -\omega\sqrt{A^2 - 0} = -\omega A$.
This is the maximum velocity which a Particle in S.H.M. Can execute and is called velocity amplitude denoted by V_{max}.
$$V_{max} = \omega A = \frac{2\pi}{T}.A$$

Case II

When the Particle is at the extreme Position, then $x = \pm A$ So.
$$V = -\omega\sqrt{A^2 - A^2} = -\omega 0 = 0$$
Thus the velocity of a Particle in S.H.M is zero at either of its extreme Positions.

Acceleration of a Particle in S.H.M.
Consider a Particle P moving with uniform angular speed

ω in a circle of radius A. The particle has the centripetal acceleration a_c, acting radially towards the center O. The magnitude of this acceleration is $a_c = \omega^2 A$.



PP' and QQ' perpendiculars to the diameter xx', the motion of P' is S.H.

The instantenous acceleration of a particle executing S.H.M. will be

$a(t)$ = Acceleration of particle P' at any instant t

= Projection of the acceleration a_c of the reference particle P

= Projection of PQ on diameter xx'

$= P'Q' = -a_c \cos(\omega t + \phi_0)$

or $a(t) = -\omega^2 A \cos(\omega t + \phi_0)$

$= -\omega^2 x$. $\because x = A\cos(\omega t + \phi_0)$

It shows that the acceleration of a particle in S.H.M is proportional to its displacement from the mean position and acts in the opposite direction of the displacement.

Case I.

When the particle is at the mean position, then.

$x = 0$ so acceleration $= -\omega^2 \times 0 = 0$.

Case II

When the particle is at the extreme position, then-

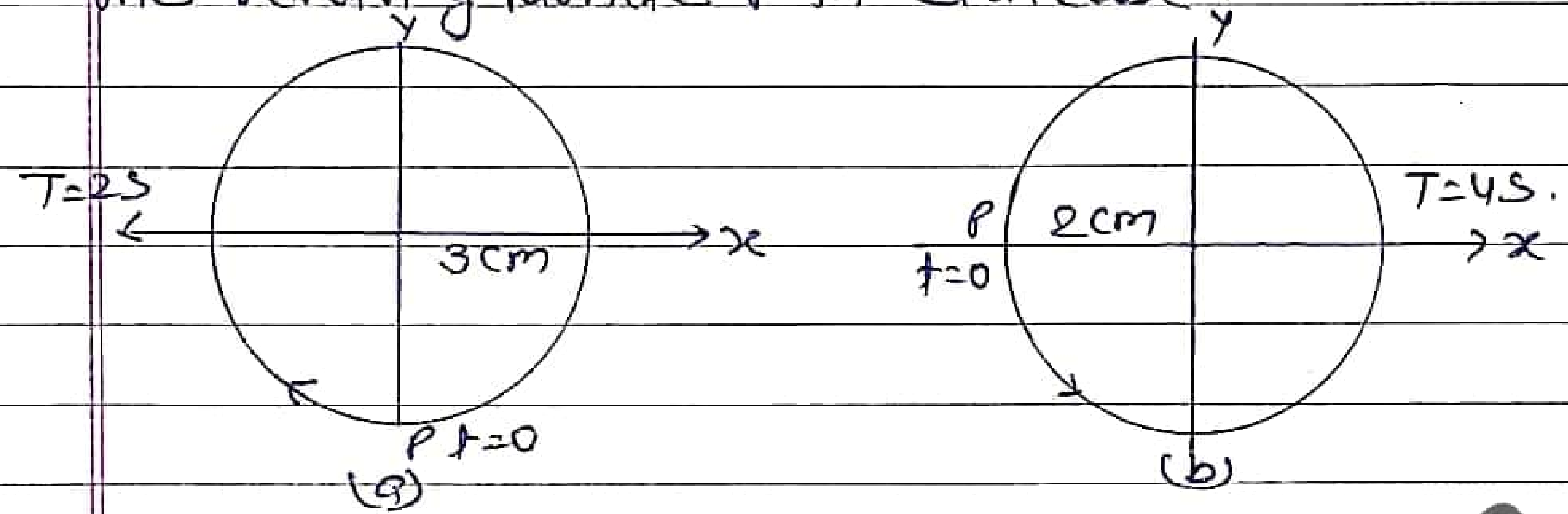
$x = A$ so acceleration $= -\omega^2 A$.

this is the maximum value of acceleration which a particle in S.H.M. can possess and is called acceleration amplitude, denoted by a_{max}.

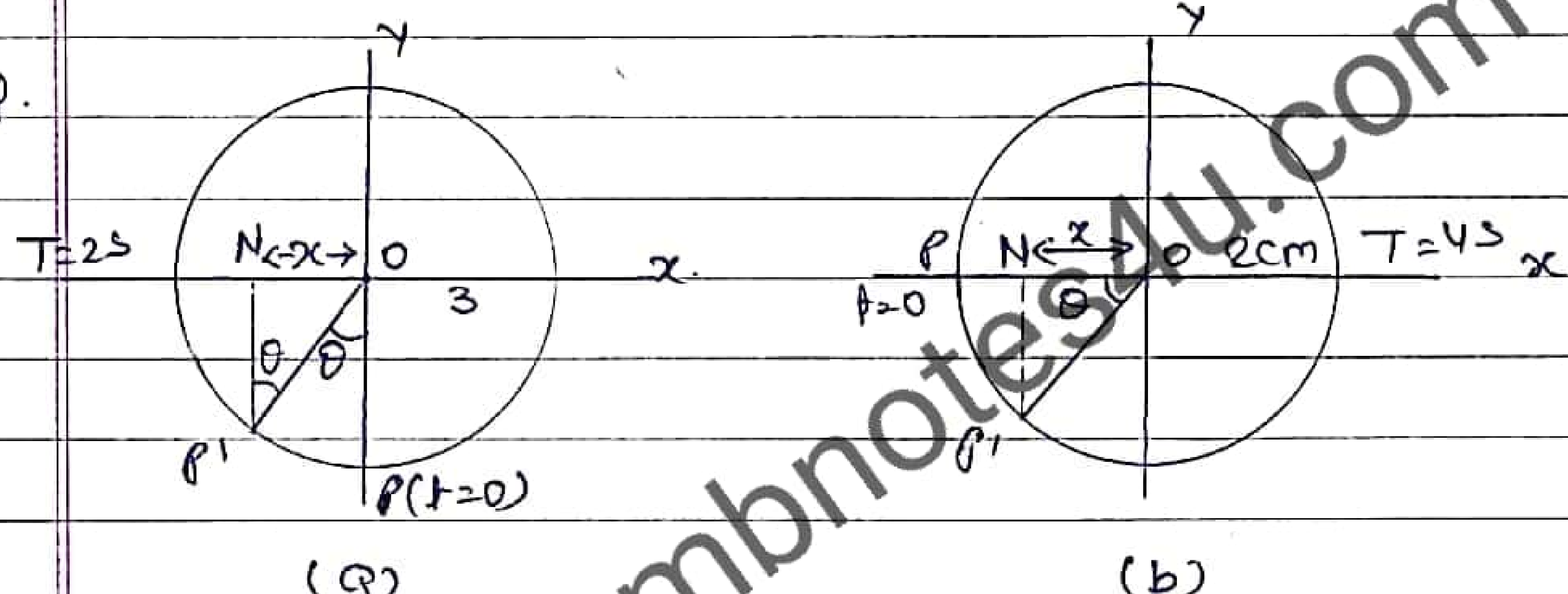
$$a_{max} = \omega^2 A = \left(\frac{2\pi}{T}\right)^2 \times a.$$

Q. The figure corresponds to two circular motions. The radius of the circle, the period of revolution, the initial position and the sence of revolution are indicated on each figure. obtain the

Corresponding S.H.M. of the x-Projection of the radius vector of the revolving Particle P in each case.



Sol.



(a) Suppose the particle moves from P to P' in time t.

$$\theta = \omega t = \frac{2\pi}{T} \times t = \frac{2\pi}{2} \times t = \pi t \text{ rad.}$$

Displacement $ON = OP' \sin\theta$

$$-x(t) = 3\sin\theta$$

$$x(t) = -3\sin \pi t \quad \text{(Displacement is left to O)}$$

(b) Suppose the particle moves from P to P' in time t.

$$\theta = \omega t = \frac{2\pi}{T} \times t = \frac{2\pi}{4} \times t = \frac{\pi t}{2}.$$

Displacment : $ON = OP' \cos\theta$

$$-x(t) = 2\cos\frac{\pi t}{2}$$

$$x(t) = -2\cos\frac{\pi t}{2}$$

Q. Plot the corresponding reference circle for each of the following S.H.M. motions.

(I) $x = -2\sin(3t + \pi/3)$ (II) $x = \cos(\pi/6 - t)$

(III) $x = 3\sin(2\pi t + \pi/4)$ (IV) $x = 2\cos\pi t$.

Sol. (I) $x = -2\sin(3t + \pi/3)$

$= 2\cos(3t + \pi/3 + \pi/2)$

$= 2\cos(5\pi/6 + 3t)$ or $= 2\cos(3t + 5\pi/6)$

Comparing with $x = A\cos(\omega t + \phi_0)$

$A = 2$ cm. $\omega = 3$ rad/sec. and $\phi_0 = 5\pi/6$ rad.

(II) $x = \cos(\pi/6 - t) = \cos[-(t - \pi/6)] = \cos(t - \pi/6)$.

then $A = 1$ cm. $\omega = 1$ rad/sec. $\phi_0 = -\pi/6$ rad.

(III) $x = 3\sin(2\pi t + \pi/4)$

$= -3\cos(2\pi t + \pi/4 + \pi/2)$

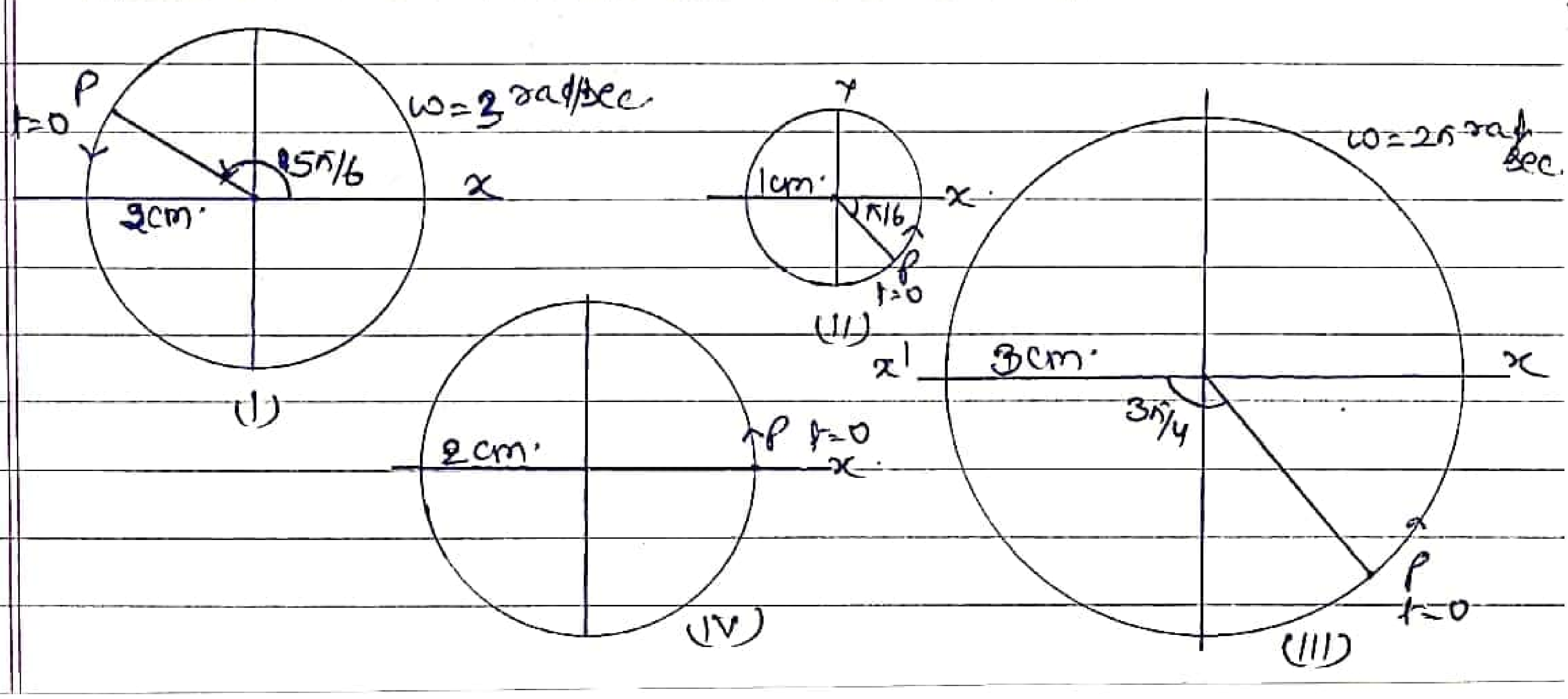
$= -3\cos(2\pi t + 3\pi/4)$.

The negative shows that the motion starts on the negative side of the x axis.

$A = 3$ cm $\omega = 2\pi$ rad/sec. $\phi_0 = 3\pi/4$ rad.

(IV) $x = 2\cos\pi t$.

$A = 2$ cm. $\omega = \pi$ rad/sec. $\phi_0 = 0$ rad.



Q. A Particle in S.H.M. is describe by the displacement function-

$$x(t) = A\cos(\omega t + \phi) \qquad \omega = \frac{2\pi}{T}.$$

If the initial ($t=0$) Position of the Particle is 1 cm and its initial velocity is π cm/sec. what are its amplitude and initial Phase angle. The angular frequency of the Particle is π sec^{-1}.

An. At $t=0$ $x = 1$ cm. and $v = \pi$ cm/sec^{-1}

$$\omega = \pi \ sec^{-1}$$

on $t=0$

$$x = A\cos(\omega t + \phi).$$

$$1 = A\cos(\omega \times 0 + \phi).$$

$$A\cos\phi = 1 \qquad —①$$

Now velocity,

$$v = \frac{dx}{dt} = -\omega A\sin(\omega t + \phi)$$

At $t=0$ $v = \pi$ cm/sec.

$$\pi = -\omega A\sin(\omega \times 0 + \phi)$$

$$\pi = -\pi A\sin\phi.$$

$$\text{then } A\sin\phi = -1 \qquad —②$$

By sq. ① and ② and then add.

$$A^2\cos^2\phi + A^2\sin^2\phi = 1 + 1$$

$$A^2 = 2$$

$$A = \sqrt{2}.$$

Dividing ② by ①

$$\frac{A\sin\phi}{A\cos\phi} = \frac{-1}{1}$$

$$\tan\phi = -1 = -\tan\pi/4 = \tan\frac{3\pi}{4} \text{ or } \tan\frac{7\pi}{4}$$

$$\phi = \frac{3\pi}{4} \text{ or } \frac{7\pi}{4}.$$

Q. The Piston in a cylinder head of a locomotive has a stroke (twice the amplitude) of 1.0 m. If the Piston moves with S.H.M. with an angular frequency 200 rev/min. what is the maximum speed.

Here $A = \frac{1}{2}$ m. $\omega = 200$ rev/min.

$$V_{max} = \omega A = 200 \times \frac{1}{2} = 100 \text{ m/min.}$$

— Energy in S.H.M : kinetic and Potential energies.

Total energy in S.H.M – The energy of a harmonic oscillator is Partly kinetic and Partly Potential. When a body is displaced from its equilibrium Position by doing work upon it, it acquire Potential energy. When the body is release it begins to move back with a velocity, thus acquiring kinetic energy.

a. kinetic energy – At any instant, the displacement of a Particle executing S.H.M. is given by.

$$x = A\cos(\omega t + \phi_0)$$

$$\text{velocity } v = \frac{dx}{dt} = -\omega A \sin(\omega t + \phi_0)$$

Hence kinetic energy of a Particle at any displacement x is

$$K = \frac{1}{2}mv^2 = \frac{1}{2}m\omega^2 A^2 \sin^2(\omega t + \phi_0)$$

But $A^2\sin^2(\omega t + \phi_0) = A^2[1 - \cos^2(\omega t + \phi_0)]$
$= A^2\left[1 - \frac{x^2}{A^2}\right] = A^2 - x^2.$

then $K = \frac{1}{2}m\omega^2(A^2 - x^2)$

b. Potential energy: when the displacement of the Particle from its equilibrium Position is x then the restoring force acting on it is $F = -kx$.

If we displace the Particle further though a small distance dx. then work done against the restoring force is given by $dW = -F\,dx = kx\,dx$.

The total work done in moving the body from mean Position $x=0$ to displacement x is

$$W = \int dW = \int_0^x F \cdot dx = \int_0^x kx \cdot dx = \frac{kx^2}{2}$$

This work done is stored as the Potential energy of the Particle.

$$U = \frac{1}{2}kx^2 = \frac{1}{2}m\omega^2x^2 \qquad \left[\because \omega^2 = \frac{k}{m}\right]$$

$$= \frac{1}{2}m\omega^2A^2\cos^2(\omega t+\phi)$$

C. Total energy: At any displacement x, the total energy of a harmonic oscillator is given by.

$$E = K + U = \frac{1}{2}k(A^2-x^2) + \frac{1}{2}kx^2.$$

$$= \frac{1}{2}kA^2 = \frac{1}{2}m\omega^2A^2.$$

$$= 2\pi^2 m \nu^2 A^2 \qquad [\because \omega = 2\pi\nu]$$

Thus the total mechanical energy of a harmonic oscillator is independent of time or displacement

Hence in the absence of any frictional force, the total energy of a harmonic oscillator is conserved.

The total energy of Particle in S.H.M is.

i. directly Proportional to the mass m of the Particle.

ii) directly Proportional to the square of its frequency ν

iii) directly Proportional to the square of its vibrating Amplitude.

Q. Show that a Particle in linear S.H.M, the average K.E. over a Period of oscillation equals the average Potential energy over the same Period.

An. Suppose a Particle of mass m executes S.H.M. of Period T. The displacement of the Particle at any instant t is

$$y = A\sin\omega t.$$

Velocity $v = \frac{dy}{dt} = \omega A\cos\omega t.$

$K.E = \frac{1}{2}mv^2 = \frac{1}{2}m\omega^2 A^2 \cos^2\omega t.$

$P.E = \frac{1}{2}m\omega^2 y^2 = \frac{1}{2}m\omega^2 A^2 \sin^2\omega t.$

Average K.E. over a Period of oscillation.

$$K_{av} = \frac{1}{T}\int_0^T K.dt = \frac{1}{T}\int_0^T \frac{1}{2}m\omega^2 A^2 \cos^2\omega t.dt..$$

$$= \frac{1}{2T}m\omega^2 A^2 \int_0^T \frac{1+\cos 2\omega t}{2}dt.$$

$$= \frac{1}{4T}m\omega^2 A^2 \left[t + \frac{\sin 2\omega t}{2\omega}\right]_0^T$$

$$= \frac{1}{4T}m\omega^2 A^2 \left[T + \frac{\sin 2\omega T}{2\omega}\right] \quad (\omega T = 2\pi)$$

then $\frac{1}{4T}m\omega^2 A^2 \left[T + \frac{\sin 4\pi}{2\omega}\right]$

$$\frac{1}{4T}m\omega^2 A^2 (T) = \frac{1}{4}m\omega^2 A^2 \text{ —(I).} \quad \because \sin 4\pi = 0$$

Av. Potential energy.

$$P_{av} = \frac{1}{T}\int_0^T \frac{1}{2}m\omega^2 A^2 \sin^2\omega t.$$

$$= \frac{1}{2T}m\omega^2 A^2 \int_0^T \frac{1-\cos 2\omega t}{2}$$

$$= \frac{1}{4T}m\omega^2 A^2 \left[t - \frac{\sin 2\omega t}{2\omega}\right]_0^T$$

$$= \frac{1}{4T}m\omega^2 A^2 \left[T - \frac{\sin 2\omega \cdot T}{2\omega}\right]$$

$$= \frac{1}{4T}m\omega^2 A^2 [T - 0] = \frac{1}{4}m\omega^2 A^2 \quad \text{—(II)}$$

then $K_{Av} = P_{Av}.$

Oscillations Due to Spring.

Consider a massless spring lying on a frictionless horizontal table. Its one end is attached to a rigid support and the other end to a body of mass m, If the body is pulled towards right through a small distance x and released. it starts oscillating back and forth about its equilibrium position under the action of the restoring force of elasticity.

$$F = -kx.$$

Where k is the force constant of the spring. The negative sign indicates that the force is directed oppositely to x.

If $\frac{d^2x}{dt^2}$ is the acceleration of the body, then

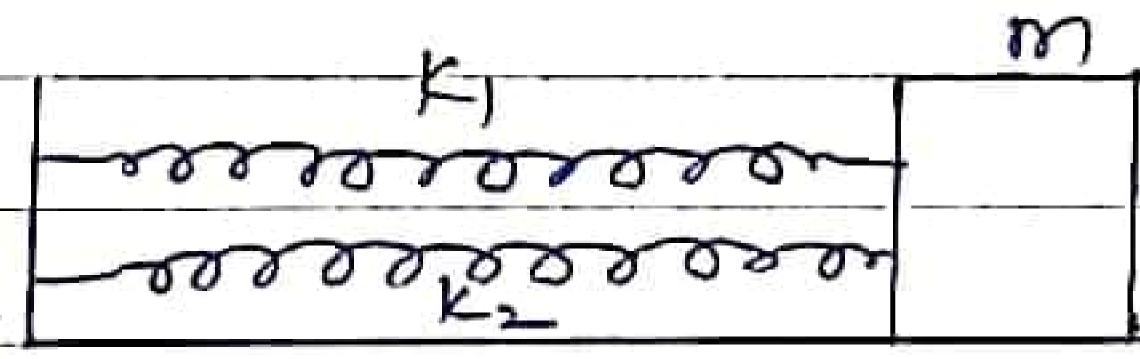
$$m \cdot \frac{d^2x}{dt^2} = -kx.$$

$$\text{then } \frac{d^2x}{dt^2} = -\frac{k}{m}x. = -\omega^2 x. \qquad \because \omega^2 = \frac{k}{m}.$$

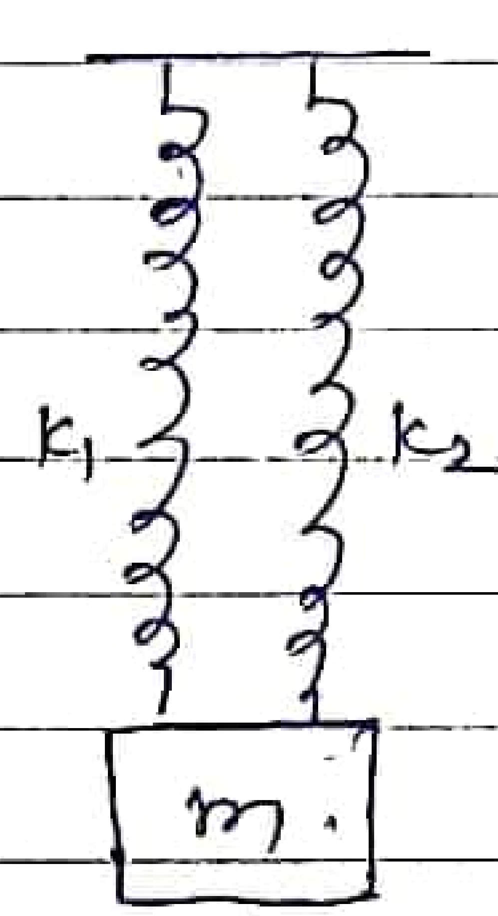
Time Period. $T = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{m}{k}}.$

Q. Fig. shows four different spring arrangements. If the mass of each arrangement is displaced from its equilibrium position and released, what is the resulting frequency of vibration in each case. Neglect the mass of each spring.

An. Spring Connected in Parallel.



Two spring Connected in Parallel.
Let k_1 and k_2 be their spring constant
Let y be the extension produced in each spring
Restoring force produced in the two springs will be $F_1 = -k_1 y$ and $F_2 = -k_2 y$



Total Restoring force is.

$$F = F_1 + F_2 = -(k_1 + k_2)y$$

Let k_p be the force constant of the Parallel Combination.

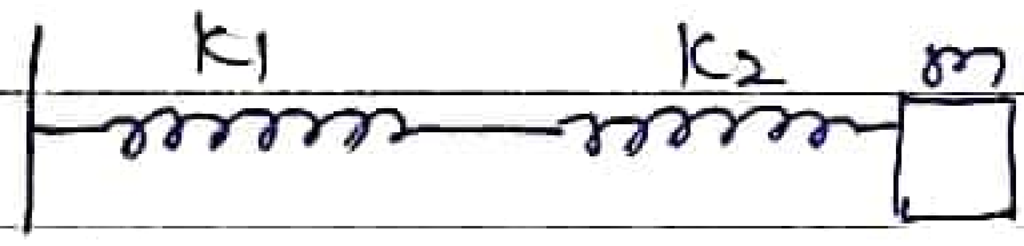
then $F = -k_p y$.

where $k_p = k_1 + k_2$

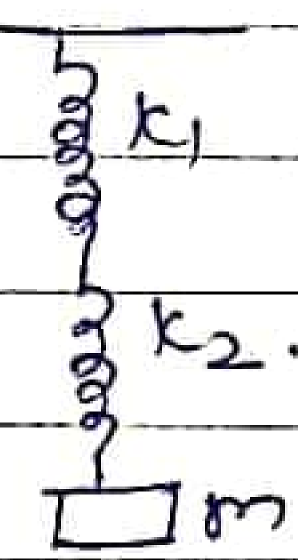
Frequency of vibration of the Parallel Combination is.

$$\nu_p = \frac{1}{2\pi}\sqrt{\frac{k_p}{m}} = \frac{1}{2\pi}\sqrt{\frac{k_1+k_2}{m}}$$

(ii) Springs Connected in series.



Let x_1 and x_2 be the extension Produced in the two springs. The restoring force F acting in the two springs is same.



$$\therefore F = -k_1 x_1 = -k_2 x_2.$$

$$x_1 = \frac{F}{-k_1} \quad \text{and } x_2 = \frac{F}{-x_2}$$

then total extension $x = x_1 + x_2 = \frac{F}{-k_1} + \frac{F}{-k_2} = -F\left(\frac{k_1+k_2}{k_4 k_2}\right)$

$$\therefore F = -\frac{k_1 k_2}{k_1+k_2} x$$

Let k_s be the force constant of the series Combination.

$\therefore F = -k_s x$. where $k_s = \frac{k_1 k_2}{k_1+k_2}$.

Frequency of oscillation of the series Combination is.

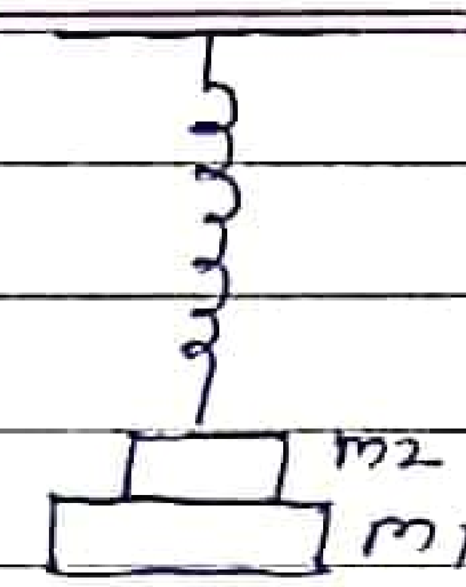
$$\nu_s = \frac{1}{2\pi}\sqrt{\frac{k_s}{m}} = \frac{1}{2\pi}\sqrt{\frac{k_1 k_2}{m(k_1+k_2)}}$$

Q. Two masses m_1 and m_2 are suspended together by a massless spring of force constant k, when they are in equilibrium position, m_1 is gently removed. Calculate angular frequency and the amplitude of oscillation of m_2.

An. Let y be the extension in the length of the spring when both both m_1 and m_2 are suspended then.

$F = (m_1+m_2)g = ky.$

or $y = \frac{(m_1+m_2)g}{k}.$



Let extension be y' when m_1 is removed then.

$m_2 g = ky'$ or $y' = \frac{m_2 g}{k}.$

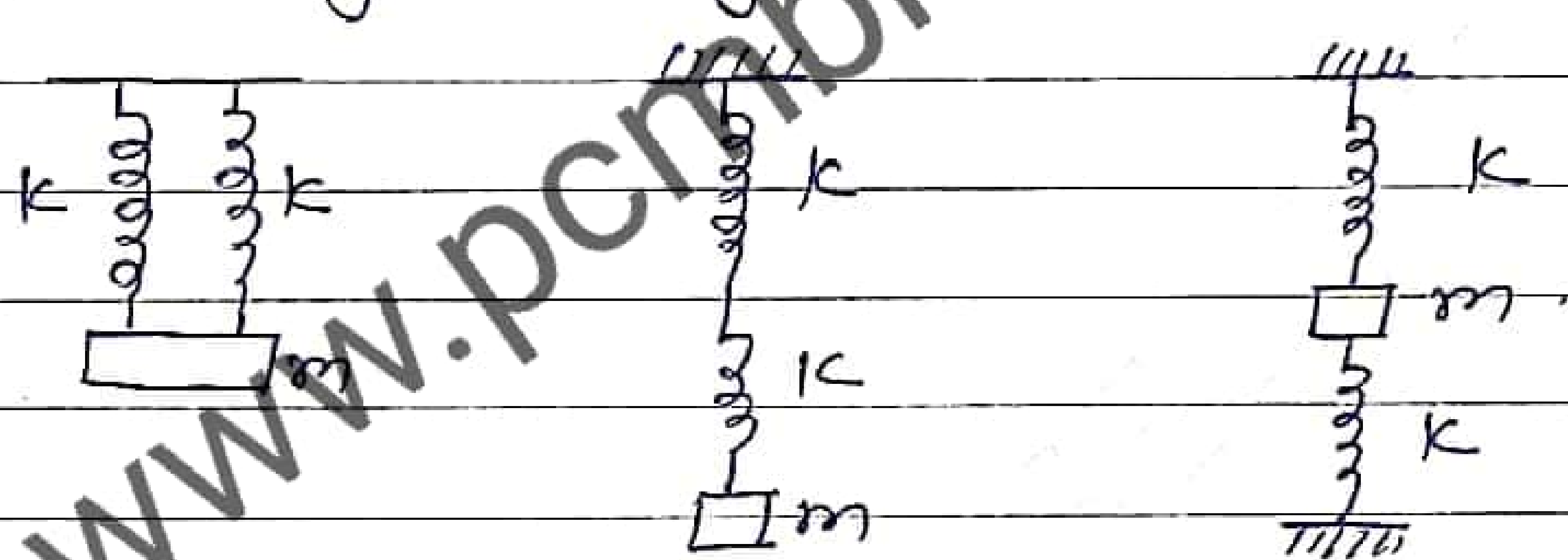
$\therefore y - y' = \frac{(m_1+m_2)g}{k} - \frac{m_2 g}{k} = \frac{m_1 g}{k}.$

This will be the amplitude of oscillation of m_2

Amplitude $A = \frac{m_1 g}{k}$

Angular frequency $\omega = \sqrt{\frac{k}{m_2}}.$

Q. Two identical springs, each of spring factor k may be connected in the following ways. Deduce the spring factor of the oscillation of the body in each case.



Ans. For each spring $F = -ky.$ —①

where F = restoring force, k = spring factor and y = displacement of the spring.

(i) Let the mass m produce a displacement y in each spring and F be the restoring force in each spring. If k_1 be the spring factor of the combined system then.

$2F = -k_1 y$ or $F = -\frac{k_1}{2} y.$ —②.

on comparing ① and ②.

$k = \frac{k_1}{2}$ or $k_1 = 2k.$

(II) As the length of the spring is double the displacement (2Y) If k_2 be the spring factor of the combined system.
then

$$F = -k_2(2Y) = -2k_2 Y \quad — ③$$

on Comparing ① and ③.

$$2k_2 = k \quad \text{or} \quad k_2 = \frac{k}{2}.$$

(III) The mass m streached the upper spring and compress the lower spring, each giving rise to a restoring force F in the same direction. If k_3 be the spring factor of the combined system. then.

$$2F = -k_3 Y \quad \text{or} \quad F = -\frac{k_3}{2} Y \quad — ④$$

on Comparing ① and ④.

$$\frac{k_3}{2} = k \quad \text{or} \quad k_3 = 2k.$$

Q. (a) Fig shows a spring of force constant k clamped rigidly at one end and a mass m attached to its free end. The spring is streached by a force F at its free end.
(b) Show the same spring with both ends free and attached to a mass m at either end. Each end of the spring is streached by the same force F.

(a) (b)

(I) What is the maximum extension of the spring in two cases. (II) If the mass in (a) and the two masses in (b) are released free. What is the period of oscillation in each case.

Ans (I) Maximum extension of the spring — In case (b), the force at either end of the spring is F and they act in opposite directions. In case (a) the force of reaction at the clamp end is also F, so both system are identical.

The maximum extension in each case is.

$$y = \frac{F}{k}.$$

(ii) In case (a) the period of oscillation is.

$$T = 2\pi\sqrt{\frac{m}{k}}.$$

In case (b) the spring can be considered to be divided into two equal halves and its center can be regarded to be fixed as it doesnot moves. let k' be the force constant of each half and x' be the extension produced in each half. then.

$$x' = \frac{F}{k'}$$

Total extension: $x = 2x'$

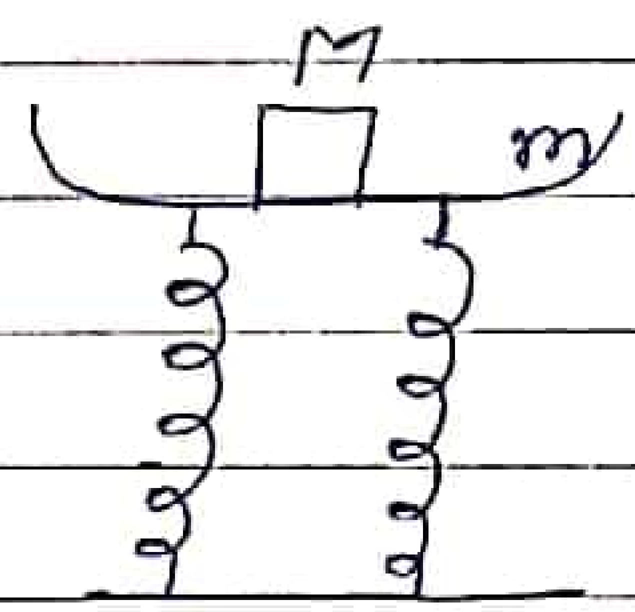
$$\frac{F}{k} = 2 \cdot \frac{F}{k'}$$

$$\therefore k' = 2k$$

Hence the period of oscillation in case (b) is

$$T = 2\pi\sqrt{\frac{m}{k'}} = 2\pi\sqrt{\frac{m}{2k}}.$$

Q. A tray of mass m is suspended by two identical springs as shows in figure. when the tray is pressed slightly and released. It excutes S.H.M. with a time period of T. what is the force constant of each spring. when a block of mass M is placed on the tray, the period of S.H.M. change to T', what is the mass of block.



Ans. Let k be the force constant of each spring. As the two springs are connected in parallel so the force constant of the combination is.

$$k' = k + k = 2k.$$

Now $T = 2\pi\sqrt{\frac{m}{k'}}$ or $k' = \frac{4\pi^2 m}{T^2}$

when the block of mass M is placed in the tray, the period of oscillation becomes

$$T' = 2\pi\sqrt{\frac{M+m}{k'}}$$

then $\frac{T'}{T} = \sqrt{\frac{M+m}{m}}$ or $\frac{M+m}{m} = \left(\frac{T'}{T}\right)^2$.

$$M = m\left[\left(\frac{T'}{T}\right)^2 - 1\right]$$

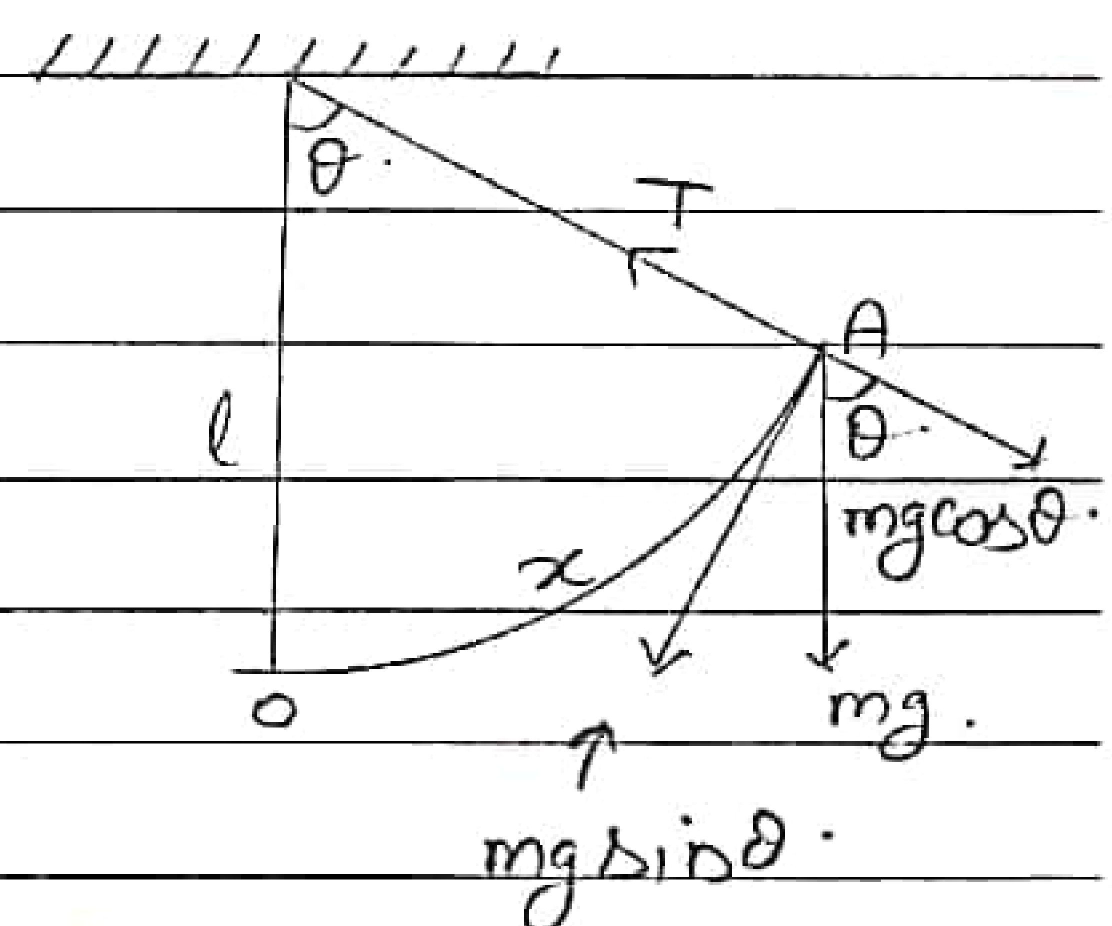
— Simple Pendulum — An ideal simple pendulum consist of a point-mass suspended by a flexible, inelastic and weightless string from a rigid support of infinite mass. In practice, we can neither have a point mass nor a weightless string then.

A simple pendulum is obtained by suspending a small metal bob by a long and fine cotton thread from a rigid support.

— Expression for time period.

In the equilibrium position, the bob of a simple pendulum lies vertically below the point of suspension. If the bob is slightly displaced on either side and released, it begins to oscillate about the mean position.

Suppose at any instant during oscillation, the bob lies at position A. when its displacement is OA = x and the thread makes angle θ with the vertical. The force acting on the bob are.



1. Weight mg of the bob downward
2. Tension T along the string.

The force mg has two rectangular component.

(i) The component $mg\cos\theta$ acting along the thread is balance by the tension T.

(ii) The tangential component $mg\sin\theta$ is the net force acting on the bob and tends to bring it back to the mean position. thus restoring force.

$$F = -mg\sin\theta = -mg\left(\theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \cdots\right)$$

$$= -mg\theta\left(1 - \frac{\theta^2}{3!} + \frac{\theta^4}{5!} - \cdots\right)$$

oscillations are not S.H. because the restoring force F is not proportional to the angular displacement.

However, If θ is so small that its higher powers can be neglected so

$$F = -mg\theta.$$

If l is the length of simple pendulum then $\theta = \frac{x}{l}$.

$$F = ma = -\frac{mg x}{l} \quad \text{or} \quad a = -\frac{g}{l}x = -\omega^2 x$$

thus the acceleration of the bob is proportional to its displacement x and is directed opposite to it. Hence for small oscillators, the motion of the bob is S.H.M.

Its time period.

$$T = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{l}{g}}.$$

$$\text{Frequency } \nu = \frac{1}{T} = \frac{1}{2\pi}\sqrt{\frac{g}{l}}.$$

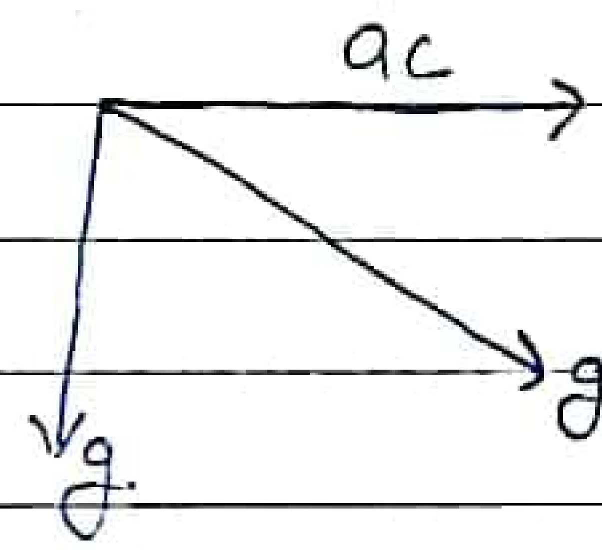
Q. A simple pendulum of length l and having a bob of mass M is suspended in a car. The car is moving on a circular track of radius R with uniform speed v. If the pendulum makes small oscillations in a radial direction about its equilibrium position. what will be its time period.

An. The bob of pendulum has two accelerations.

(i) Centripetal acceleration $a_c = \frac{v^2}{R}$ acting horizontally.

(iii) acceleration due to gravity = g acting vertically downwards. then the effective acceleration due to gravity

$$g' = \sqrt{g^2+a_c^2} = \sqrt{g^2+\frac{v^4}{R^2}}.$$



time Period $T = 2\pi\sqrt{\frac{l}{g'}} = 2\pi\sqrt{\frac{l}{g^2+\frac{v^4}{R^2}}}$

Q. A second's Pendulum is taken in a carriage. Find the Period of oscillation when the carriage moves with an acceleration of a (i) Vertically upward (ii) vertically downward (iii) Horizontal direction.

Ans. Time Period of a Pendulum.

$$T = 2\pi\sqrt{\frac{l}{g}}.$$

For seconds Pendulum $T = 2s$. → (Important)

$$2 = 2\pi\sqrt{\frac{l}{g}} \quad \text{or} \quad l = \frac{g}{\pi^2}$$

(i) when the carriage moves up ward.

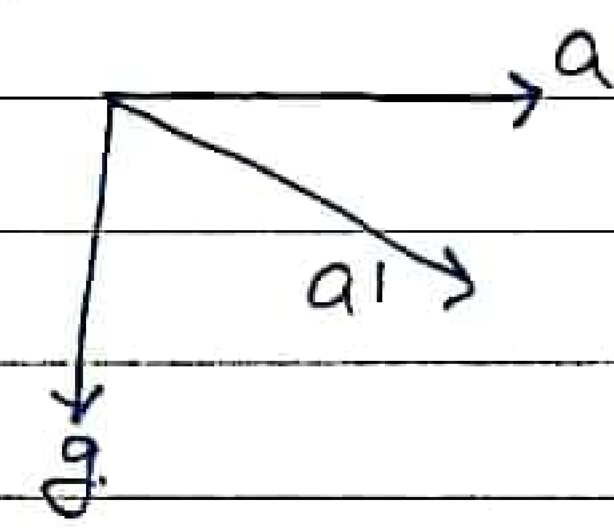
$$T_1 = 2\pi\sqrt{\frac{l}{g+a}} = 2\pi\sqrt{\frac{g}{\pi^2(g+a)}}.$$

(ii) when the carriage moves down ward.

$$T_2 = 2\pi\sqrt{\frac{l}{g-a}} = 2\pi\sqrt{\frac{g}{\pi^2(g-a)}}$$

(iii) when the carriage moves horizontally, both g and a are right angle to each other.

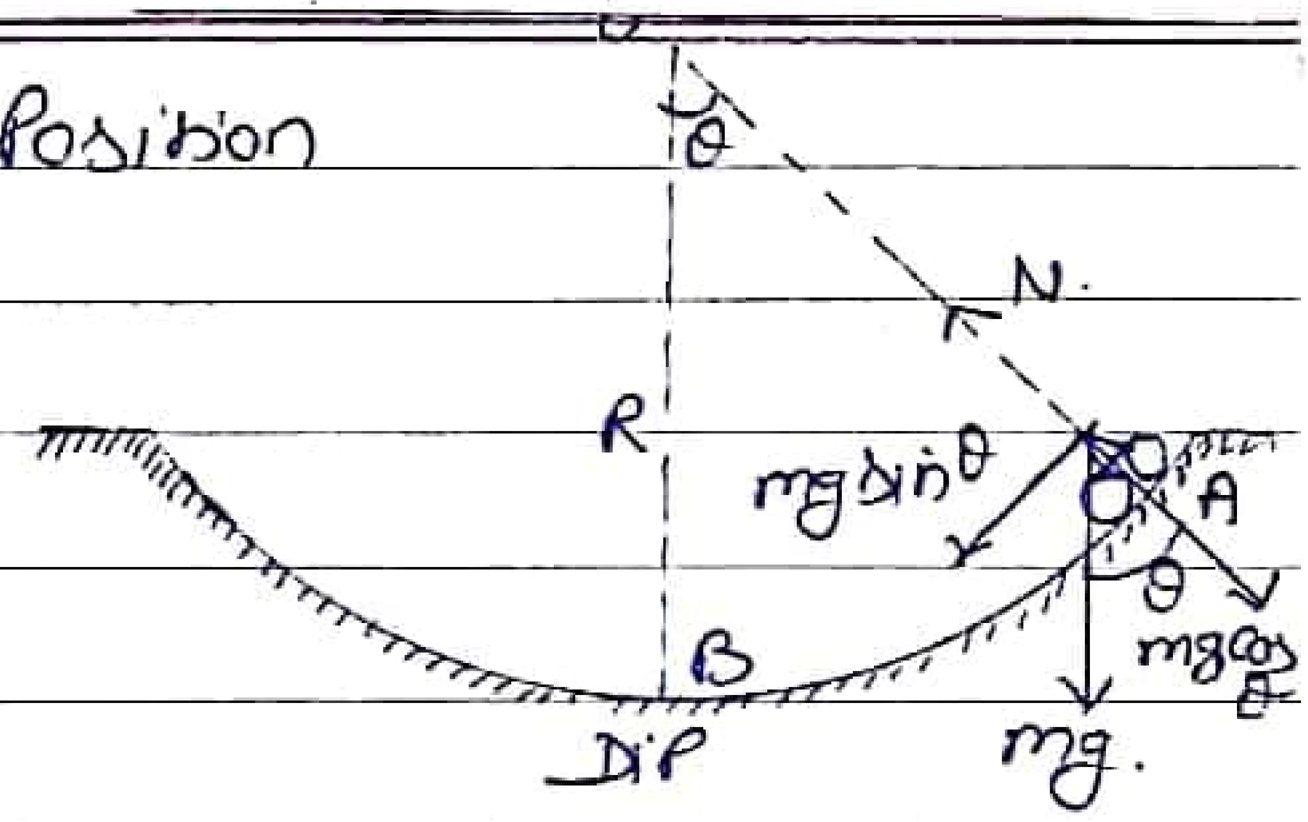
then net acceleration $a' = \sqrt{g^2+a^2}$.



$$T_3 = 2\pi\sqrt{\frac{l}{a'}} = 2\pi\sqrt{\frac{g}{\pi^2(g^2+a^2)}}$$

Q. The bottom of a dip on a road has a radius of curvature R. A Rikshaw of mass M left a little away from the bottom oscillate about the dip. Deduce an expression for the Period of oscillation.

Let the rickshaw of mass M be at position A at any instant and $\angle AOB = \theta$.



Force acting on rickshaw at position A.

(i) Weight Mg acting downward.

(ii) The normal reaction N of the Road.

The weight can be resolved into two rectangular Components.

(i) $Mg\cos\theta = N$.

(ii) $Mg\sin\theta = -F$. So $F = -mg\theta$

For small θ. (In △ OAB.) $\theta = S/r$. where $S = AB$, $r = R$.

$$\sin\theta = \theta = \frac{AB}{R} = \frac{y}{R}$$

$$F = -mg\cdot\frac{y}{R} \quad \text{or} \quad F \propto -y$$

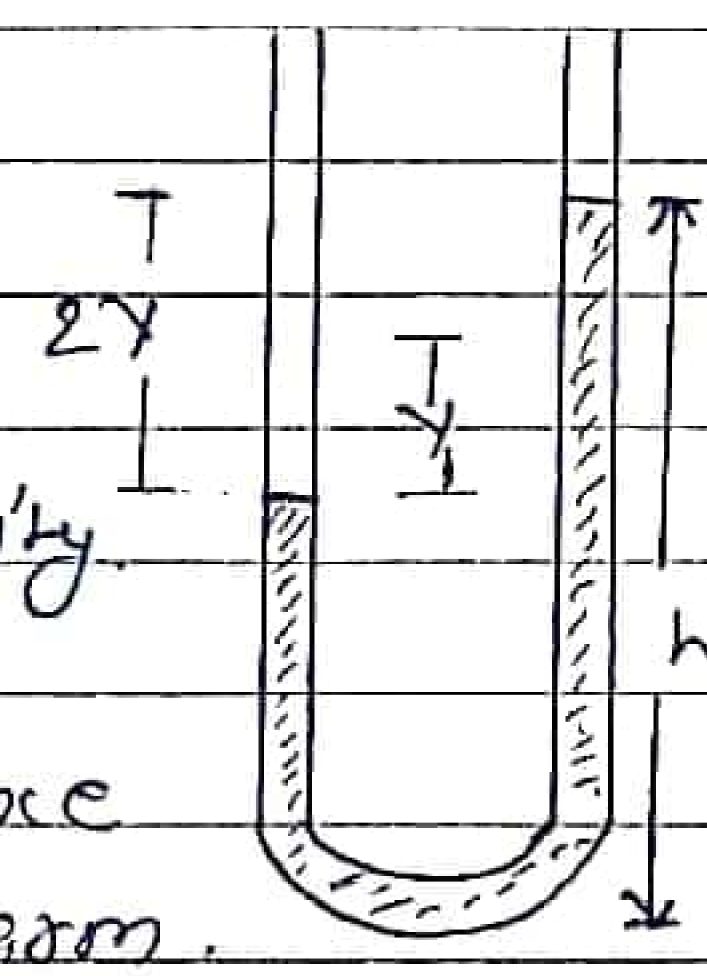
Hence the motion of the Rickshaw is S.H.M. with force constant.

$$k = \frac{Mg}{R}$$

time Period $T = 2\pi\sqrt{\frac{m}{k}} = 2\pi\sqrt{\frac{R}{g}}$

Q. One end of a U tube containing mercury is connected to a suction pump and the other end is connected to the atmosphere. A small pressure difference is maintained between the two columns. Show that when the suction pump is removed, the liquid in the U tube executes S.H.M.

Ans. Suppose the U-tube of cross section A contained liquid of density ρ upto height h.



Then mass of the liquid in U-tube.

$$m = \text{Volume} \times \text{density} = \text{Area} \times \text{height} \times \text{density}$$
$$= A \times 2h \times \rho$$

If the liquid in one arm is depressed by distance y, it rises by the same amount in the other arm. If left to itself, the liquid begins to oscillate under the weight = mass × g = Cross. Ar ... × gravity.

Restoring force.

F = weight of liquid column of height $2y$

$= -A \times 2y \times \rho \times g = -2A\rho g y.$

$F \propto -y.$

thus the force on the liquid is proportional to displacement and acts in its opposite direction. Hence the liquid in the U-tube execute S.H.M. with force constant.

$$K = 2A\rho g.$$

The time period of oscillation

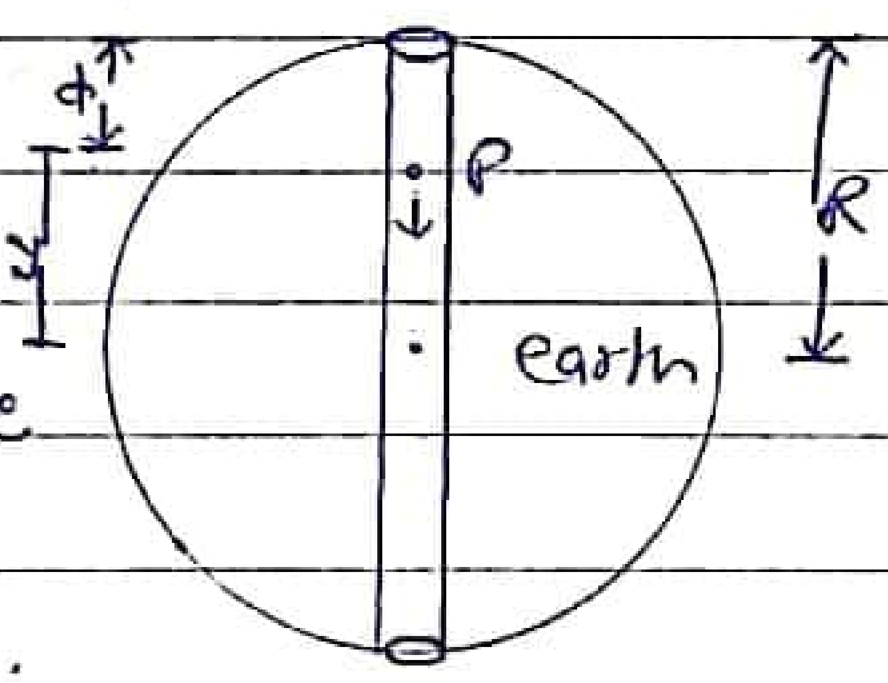
$$T = 2\pi\sqrt{\frac{m}{K}} = 2\pi\sqrt{\frac{A \times 2h \times \rho}{2A\rho g}} = 2\pi\sqrt{\frac{h}{g}}.$$

If l is the length of the liquid column then $l = 2h$.

so $T = 2\pi\sqrt{\frac{l}{2g}}.$

Q. If the earth were a homogeneous sphere of radius R and a straight hole bored in it through its center, Show that a body dropped into the hole will execute S.H.M. and find its time period.

An. Suppose a body of mass m is dropped into the tunnel and it is at point P at a depth d below the surface of the earth at any instant. If g' is acceleration due to gravity at P then:



$$g' = g\left(1 - \frac{d}{R}\right) = g\left(\frac{R-d}{R}\right)$$

If y is distance of the body from the center of the earth then. $R - d = y$ $\quad \therefore \; g' = g\frac{y}{R}.$

Force acting on the body at point P is

$$F = -mg' = -\frac{mg}{R}y. \quad \text{or} \quad F \propto y$$

Negative sign shows that the force F acts in the opposite direction of displacement. It acts towards the mean Position. Thus the body will execute S.H.M. with force constant.

$$k = \frac{mg}{R}.$$

The Period of oscillation of the body will be

$$T = 2\pi\sqrt{\frac{m}{k}} = 2\pi\sqrt{\frac{R}{g}}.$$

Q. A cylinderical Peace of Cork of base area A and height h floats in a liquid of density ρ_1. The Cork is depressed slightly and then released. Show that the cork oscillate up and down simple harmonically with a Period $T = 2\pi\sqrt{\frac{h\rho}{\rho_1 g}}$, where ρ is the density of cork.

An. Let the Cork be slightly depressed through distance Y from the equilibrium Position and left to itself. It begins to oscillate under the restoring force.

F = Net upward force

= weight of liquid column of height Y

$= -AY\rho_1 g = -A\rho_1 g Y$

$F \propto Y.$

Hence the Cork execute the S.H.M. with force constant.

$$k = A\rho_1 g.$$

then time Period.

$$T = 2\pi\sqrt{\frac{m}{k}} = 2\pi\sqrt{\frac{A\rho h}{A\rho_1 g}} = 2\pi\sqrt{\frac{\rho h}{\rho_1 g}}.$$

Q. An air chamber of volume V has a neck of area of cross section A into which a ball of mass m can move without friction. Show that when the ball pressed down through some distance and released. the ball execute S.H.M. Assume Pressure-volume variations of the air to be

(i) isothermal (ii) adiabatic.

An. If the ball is depressed by distance Y then the decrease in volume of air in the Chamber is $\Delta V = AY$.

Volume strain $= \frac{\Delta V}{V} = \frac{AY}{V}$

If Pressure P is applied to the ball, then hydrostatic stress $= P$

Bulk modulus of elasticity of air.

$$E = \frac{-P}{\Delta V/V} = \frac{-P}{AY/V} \quad \text{or} \quad P = -\frac{EA}{V}.Y$$

Restoring force $F = PA = -\frac{EA^2}{V}.Y$.

or.

$$F \propto Y.$$

Hence the ball execute S.H.M. with force constant.

$$K = \frac{EA^2}{V}$$

Time Period $T = 2\pi\sqrt{\frac{m}{K}} = 2\pi\sqrt{\frac{mV}{EA^2}}$

(i) If P-V variations are isothermal then $E = P$.

$$T = 2\pi\sqrt{\frac{mV}{PA^2}}.$$

(ii) If P-V variations are adiabatic then $E = \gamma P$.

$$T = 2\pi\sqrt{\frac{mV}{\gamma PA^2}}.$$

— Free oscillation — If a body capable of oscillation, is slightly, displaced from its position of equilibrium and left to itself, it starts oscillating with a frequency of its own. Such oscillations are called free oscillation.

— Damped oscillations :- The oscillations in which the amplitude decreases gradually with the passage of time are called Damped oscillations.

- **Maintained oscillations** — If to an oscillating system, energy is continuously supplied from outside at the same rate at which the energy is lost by it, then its amplitude can be maintained constant. Such oscillations are called maintained oscillations.
- **Forced oscillations** — When a body oscillates under the influence of an external periodic force not with its own natural frequency of the external periodic force its oscillations are said to be forced oscillations. The external agent which exerts the periodic force is called the driver and the oscillating system under consideration is called the driven body.
- **Resonant oscillations and resonance** — It is a particular case of forced oscillations in which the frequency of the driving force is equal to the natural frequency of the oscillator itself and the amplitude of oscillations is very large. Such oscillations are called resonant oscillations and phenomenon is called resonance.
- **Coupled oscillations** — A system of two or more oscillators linked together in such a way that there is mutual exchange of energy between them is called a couple oscillators. The oscillations of such a system are called couple oscillations.