Gravitation

⇒ Newton's Law of Gravitation :-

"Every particle of matter in the Universe attracts every other particle with a force that is directly proportional to the product of the masses of the particles and inversely proportional to the square of the distance between them".

$$F \propto \frac{m_1 m_2}{r^2} \quad \text{or} \quad \boxed{F = G\frac{m_1 m_2}{r^2}}$$

Here, G is a Universal Constant called Gravitational Constant whose magnitude is,

$$G = 6.67 \times 10^{-11} \ N\text{-}m^2/kg^2$$

The direction of the force F is along the line joining the two particles.

(1) Unlike the electrostatic force, it is independent of the medium between the particles.

(2) It is Conservative in nature

(3) It expresses the force between two point masses (of negligible volume). However, for

external points of spherical bodies the whole mass can be assumed to be concentrated at its centre of mass.

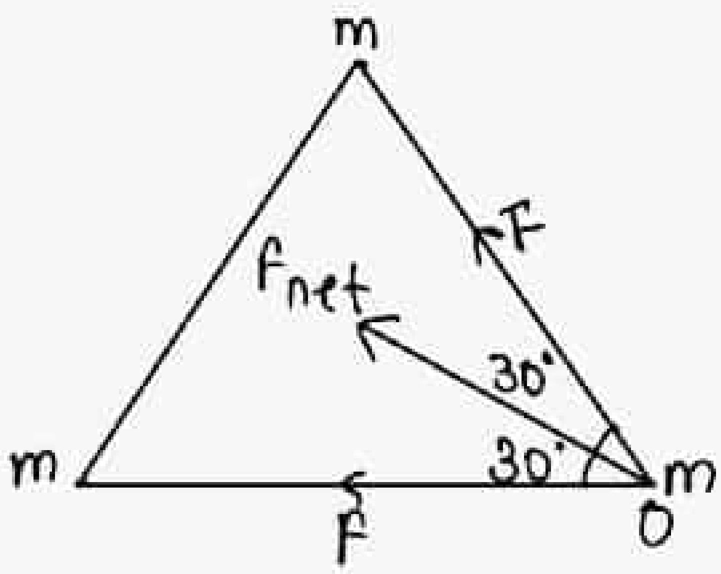
⇒ Gravity :-

In Newton's law of gravitation, gravitation is the force of attraction between any two bodies. If one of the bodies is earth then the gravitation is called 'gravity'. Hence, gravity is the force by which earth attracts a body towards its centre. It is a special case of gravitation.

Eg:- Three point masses 'm' each are placed at the three vertices of an equilateral triangle of side 'a' find net gravitational force on any one point mass.

Solⁿ→ We are finding net force on the point mass kept at O.

$$F = \frac{G(m)(m)}{a^2} = \frac{Gm^2}{a^2}$$



Since, the two forces are equal in magnitudes, therefore the resultant force will pass through the centre as shown in fig.

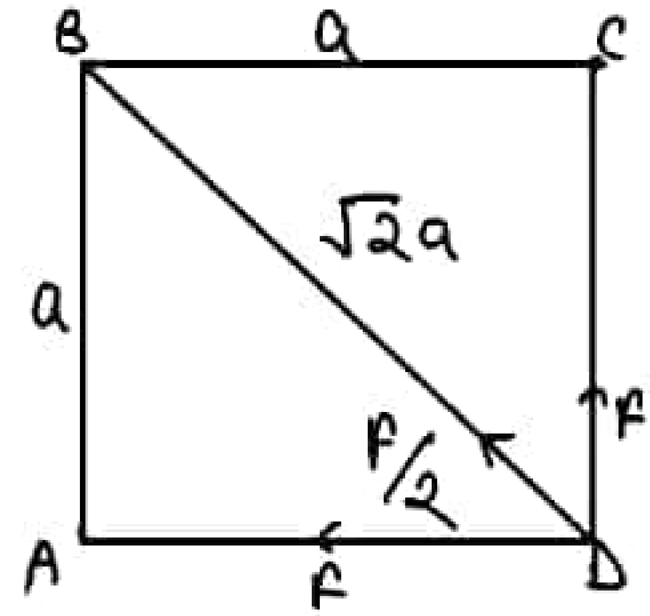
$$F_{net} = \sqrt{F^2 + F^2 + 2(F)(F)\cos 60^\circ}$$
$$= \sqrt{3}\,F$$
$$= \frac{\sqrt{3}\,Gm^2}{a^2}$$

Eg:- Four particles each of mass 'm' are placed at the four vertices of a square of side 'a'. Find net force on any one of the particle

Sol^n → We are finding net force on the particle at D.

$$F_{DC} = F_{DA} = \frac{G(m)(m)}{a^2} = \frac{Gm^2}{a^2} = F \text{ (say)}$$

$$F_{DB} = \frac{G(m)(m)}{(\sqrt{2}a)^2} = \frac{1}{2}\frac{Gm^2}{a^2} = \frac{F}{2}$$

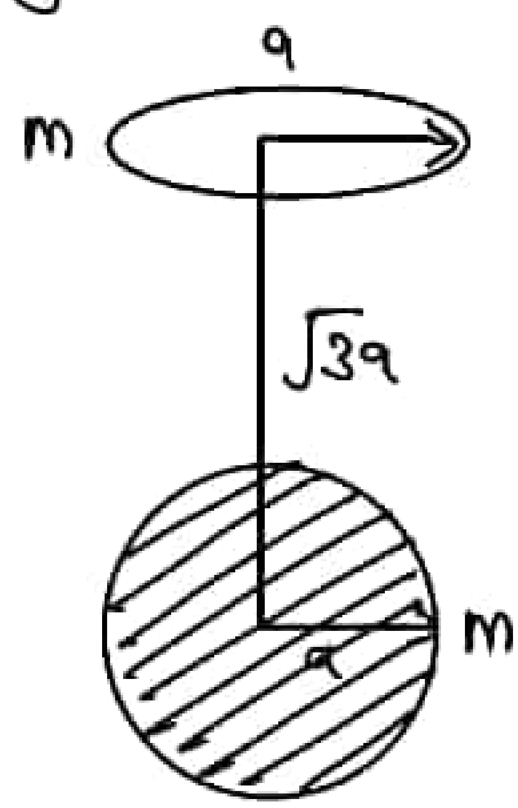


Now, resultant of F_{DA} and F_{DC} is $\sqrt{2}F$ in the direction of DB.

$$\therefore \quad F_{net} = \sqrt{2}F + \frac{F}{2} = \left(\sqrt{2}+\frac{1}{2}\right)F$$

$$= \left(\sqrt{2}+\frac{1}{2}\right)\frac{Gm^2}{a^2} \text{ (towards DB)}$$

Eg:- A uniform ring of mass m is lying at a distance $\sqrt{3}$ a from the centre of a sphere of mass M just over the sphere (where a is the radius of the ring as well as that of the sphere). Find the magnitude of gravitational force between them.

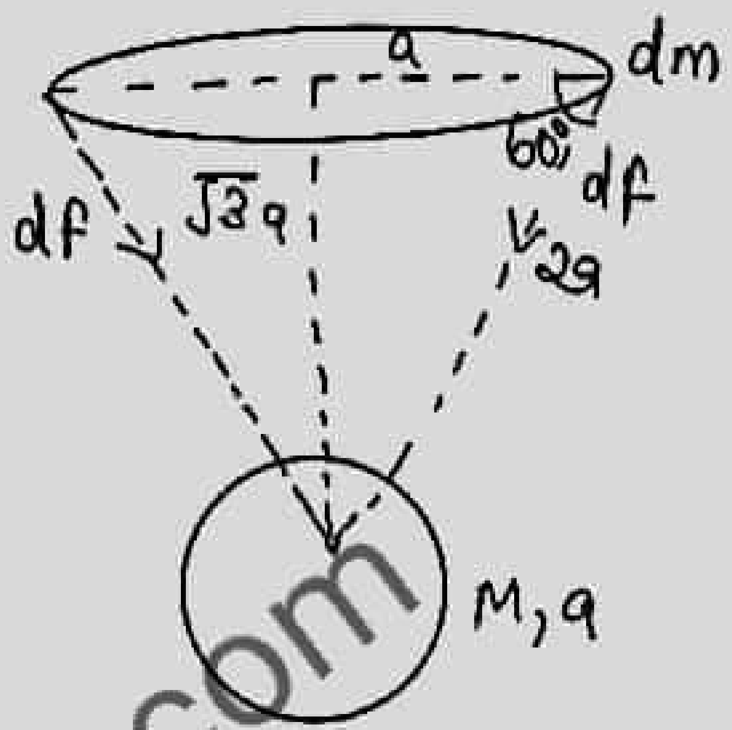


Solⁿ:- Net force on ring $= \int_{\text{whole ring}} df \sin 60^\circ$

$$= \int_{\text{whole ring}} \frac{Gm(dm)}{(2a)^2} \frac{\sqrt{3}}{2}$$

$$= \frac{\sqrt{3}\, GMm}{8a^2}$$

as $\int_{\text{whole ring}} dm = m$



Acceleration due to Gravity:-

When a body is dropped from a certain height above the ground it begins to fall towards the earth under gravity. The acceleration produced in the body due to gravity is called the acceleration due to gravity. It is denoted by g. Its value close to the earth's surface is $9.8\,m/s^2$.

Suppose that the mass of the earth is M, its radius is R, then the force of attraction acting on a body of mass m close to the surface of earth is

$$F = \frac{GMm}{R^2}$$

According to Newton's 2^{nd} law, the acceleration due to gravity

$$\boxed{g = \frac{F}{m} = \frac{GM}{R^2}}$$

Variation in the Value of g :-

The value of g varies from place to place on the surface of earth. It also varies as we go above or below the surface of earth. Thus, value of g depends on following factors:-

⇒ Shape of the Earth :-

The Earth is not a perfect sphere. It is somewhat flat at the two poles. The equatorial radius is approximately 21 km more than the polar radius and since

$$g = \frac{GM}{R^2} \quad \text{or} \quad g \propto \frac{1}{R^2}$$

The value of g is minimum at the equator and maximum at the poles

⇒ Height above the Surface of the Earth :-

The force of gravity on an object of mass m at a height h above the surface of earth is,

$$F = \frac{GMm}{(R+h)^2}$$

∴ Acceleration due to gravity at this height will be,

$$g' = \frac{F}{m} = \frac{GM}{(R+h)^2}$$

This can also be written as,

$$g' = \frac{GM}{R^2\left(1+\frac{h}{R}\right)^2}$$

or $$g' = \frac{g}{\left(1+\frac{h}{R}\right)^2}$$ as $$\frac{GM}{R^2} = g$$

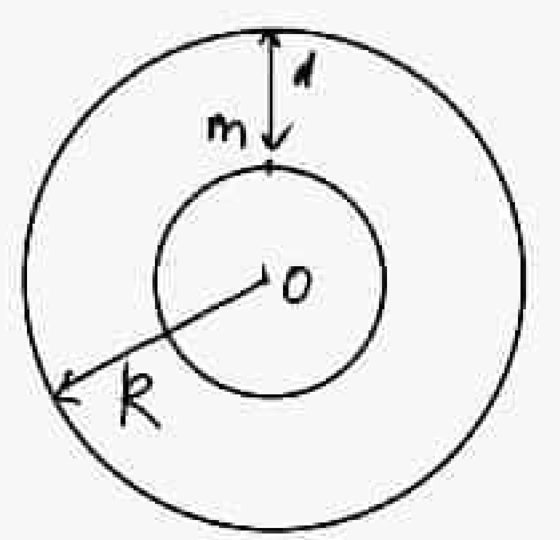
Thus, $g' < g$

i.e. the value of acceleration due to gravity g goes on decreasing as we go above the surface of earth. Further,

$$g' = g\left(1+\frac{h}{R}\right)^{-2} \quad \text{or} \quad \boxed{g' \approx g\left(1-\frac{2h}{R}\right) \text{ if } h \ll R}$$

⇒ Depth below the Surface of the Earth :-

let an object of mass m is situated at a depth d below the earth's surface. It's distance from the centre of earth is $(R-d)$. This mass is situated at the surface of the inner solid sphere and lies inside the outer spherical shell. According to Gauss theorem, the gravitational force of attraction on a mass inside a spherical shell is always zero. Therefore, the object experiences gravitational attraction only due to inner solid sphere.



The mass of this sphere is,

$$M' = \left(\frac{M}{4/3\,\pi R^3}\right)\frac{4}{3}\pi (R-d)^3$$

or, $$M' = \frac{(R-d)^3}{R^3} \cdot M$$

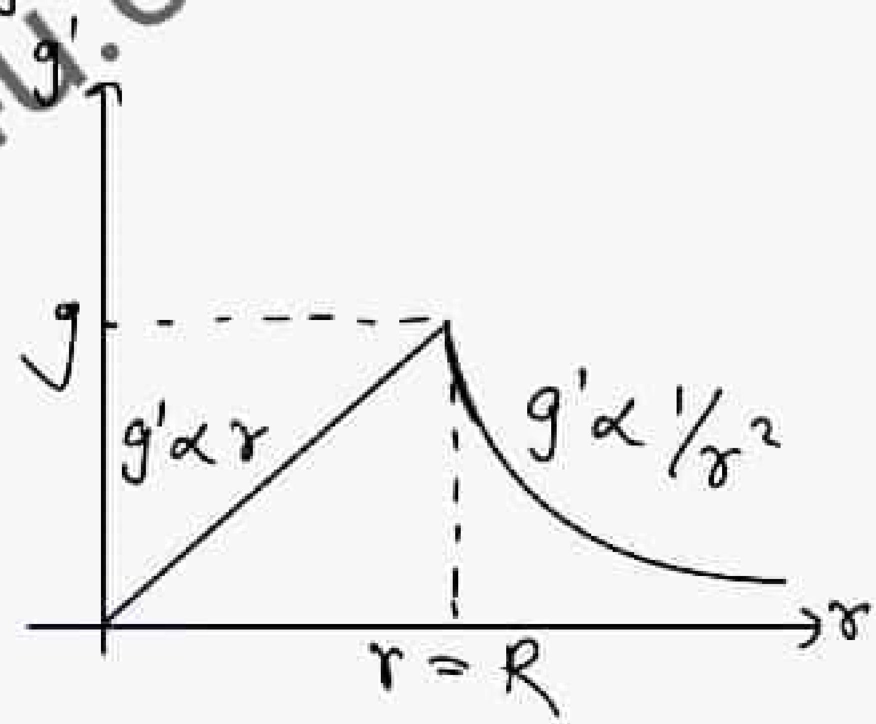
$$F = \frac{GM'm}{(R-d)^2} = \frac{GMm(R-d)}{R^3}$$

and $$g' = \frac{F}{m}$$

Substituting the values, we get $\boxed{g' = g\left(1 - \frac{d}{R}\right)}$

i.e. $g' < g$

Thus, the variation in the value of g with r (the distance from the centre of earth)



for $r \le R$, $$g' = g\left(1 - \frac{d}{R}\right) = \frac{gr}{R}$$

as $R - d = r$ or $g' \propto r$

for $r \ge R$, $$g' = \frac{g}{\left(1 + \frac{d}{R}\right)^2} = \frac{gR^2}{r^2}$$

as $\left[R + d = r \text{ or } g' \propto 1/r^2 \right]$

Axial Rotation of the Earth:-

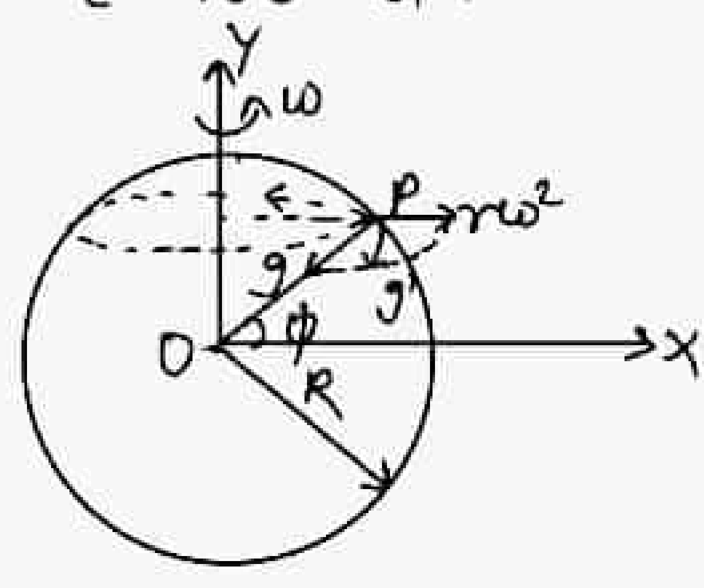
Let us consider a particle P at rest on the surface of the earth, at latitude ϕ. Then the pseudo force acting on the particle is $mr\omega^2$ in outward direction. The true acceleration g is acting towards the centre O of the earth. Thus, the effective acceleration g' is the resultant of g and $r\omega^2$ or

$$g' = \sqrt{g^2 + (r\omega^2)^2 + 2g(r\omega^2)\cos(180° - \phi)}$$

as angle between g and $r\omega^2$ is $(180° - \phi)$.

or $$g' = \sqrt{g^2 + r^2\omega^4 - 2gr\omega^2 \cos\phi} \quad \text{—— (1)}$$

Here, the term $r^2\omega^4$ comes out to be too small as $\omega = \frac{2\pi}{T} = \frac{2\pi}{24 \times 3600}$ rad/s



is small. Hence, this term can be ignored. Also, $r = R\cos\phi$. Therefore, Eq^n (1), can be written as,

$$g' = (g^2 - 2gR\omega^2 \cos^2\phi)^{1/2}$$
$$= g\left(1 - \frac{2R\omega^2\cos^2\phi}{g}\right)^{1/2}$$
$$\approx g\left(1 - \frac{R\omega^2\cos^2\phi}{g}\right)$$

Thus, $\boxed{g' = g - R\omega^2\cos^2\phi}$

Following conclusions can be drawn from the above discussion:-

(1) The effective value of g is not truly vertical passing through the centre O.

(2) The effect of centrifugal force due to rotation of earth is to reduce the effective value of g.

(3) At equator $\phi = 0°$,

Therefore, $g' = g - R\omega^2$

and at poles $\phi = 90°$,

Therefore, $g' = g$

Thus, at equator g' is minimum while at poles g' is maximum.

Eg:- Suppose the earth increases its speed of rotation. At what new time period will the weight of a body on the equator becomes zero? Take $g = 10 m/s^2$ and radius of earth $R = 6400$ km.

Solⁿ:- The weight will become zero, when

$$g' = 0 \quad \text{or} \quad g - R\omega^2 = 0 \quad (\text{on the equator } g' = g - R\omega^2)$$

or $$\omega = \sqrt{\frac{g}{R}}$$

$$\therefore \quad \frac{2\pi}{T} = \sqrt{\frac{g}{R}} \quad \text{or} \quad T = 2\pi\sqrt{\frac{R}{g}}$$

Substituting the values, $$T = \frac{2\pi\sqrt{\frac{6400 \times 10^3}{10}}}{3600} \, h$$

or $$T = 1.4 \, h \quad (\text{Ans})$$

Thus, the new time period should be 1.4 h instead of 24 h for the weight of a body to be zero on the equator.

Gravitational field and field Strength :-

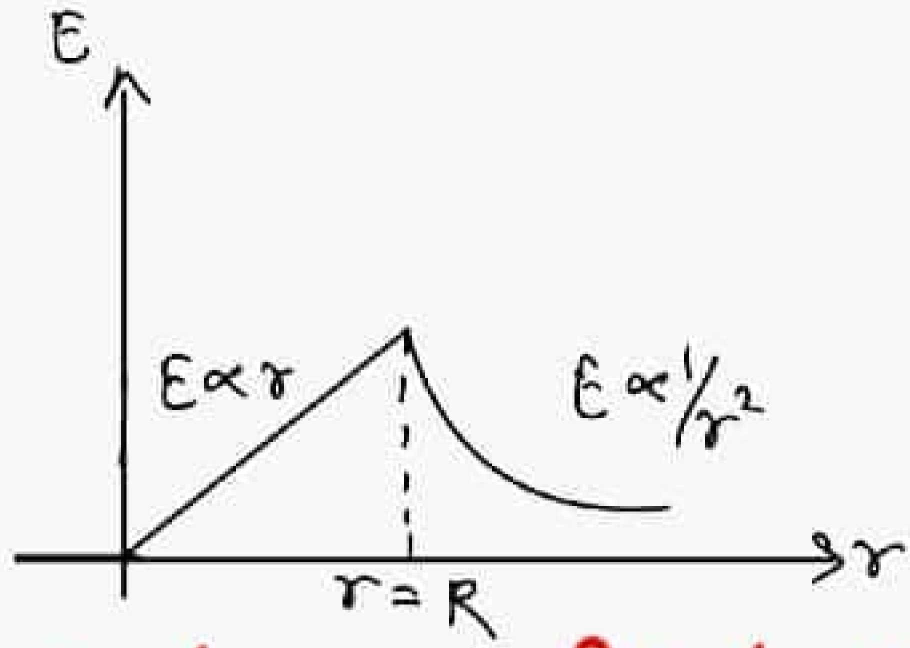
The space around a mass or system of masses in which any other test mass experiences a gravitational force is called Gravitational field.
When this test mass is moved from one point to another point, some work is also done by this Gravitational force.

Gravitational field Strength (E) :-

The force experienced (both in magnitude and direction) by a unit test mass placed at a point in a gravitational field is called the Gravitational field strength or intensity of Gravitational field at that point.

Gravitational field Strength ← $\boxed{E = \frac{F}{m}}$ → Force, → mass

S.I. Unit of E is N/Kg.



Field due to a Point Mass:-

$$F = \frac{GMm}{r^2}$$

$$E = \frac{F}{m} \quad \text{or} \quad \boxed{E = \frac{GM}{r^2}}$$

Gravitational Field due to a Uniform Solid Sphere:-

⇒ Field at an External Point:-

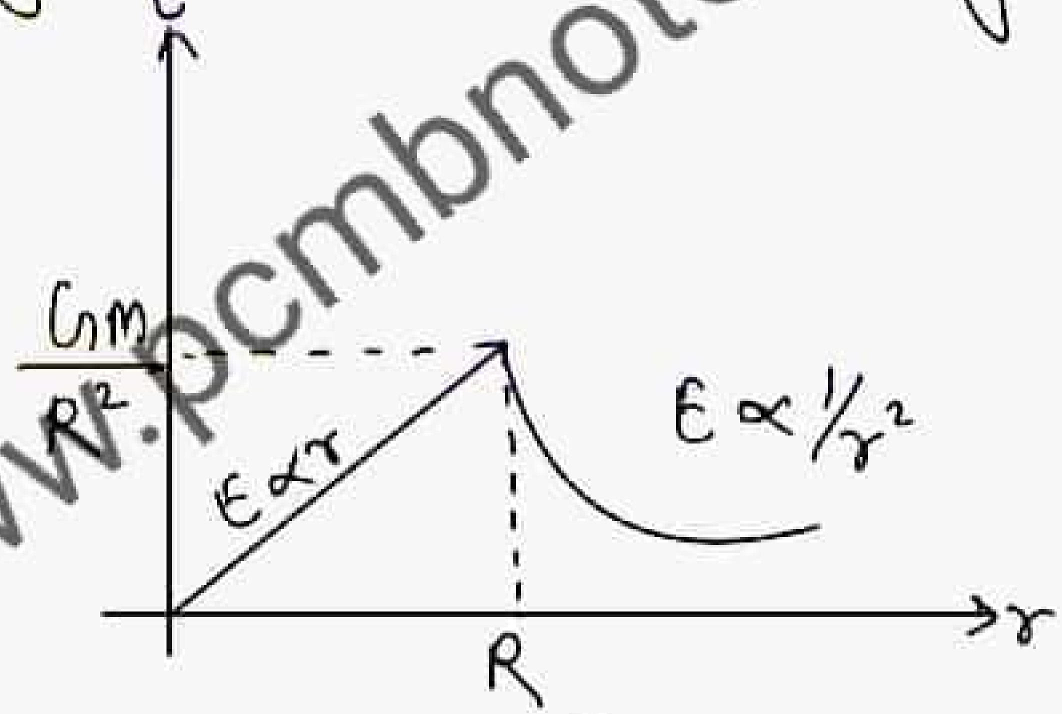
A uniform sphere may be treated as a single particle of same mass placed at its centre for calculating the gravitational field at an external point. Thus,

$$\boxed{E(r) = \frac{GM}{r^2}} \quad \text{for } r \geq R \text{ or } E(r) \propto 1/r^2$$

Here, r is the distance of the point from the centre of the sphere and R the radius of sphere.

⇒ Field at an Internal Point :-

The gravitational field due to a uniform sphere at an internal point is proportional to the distance of the point from the centre of the sphere.



At the centre itself, it is zero and at surface it is $\frac{GM}{R^2}$, where R is the radius of the sphere. Thus,

$$\boxed{E(r) = \frac{GM}{R^3} r} \quad \text{for } r \leq R \text{ or } E(r) \propto r$$

Field due to a Uniform Spherical Shell :-

⇒ At an External Point :-

For an external point the shell may be treated as a single particle of same mass placed at its centre. Thus, at an external point the gravitational field is given by,

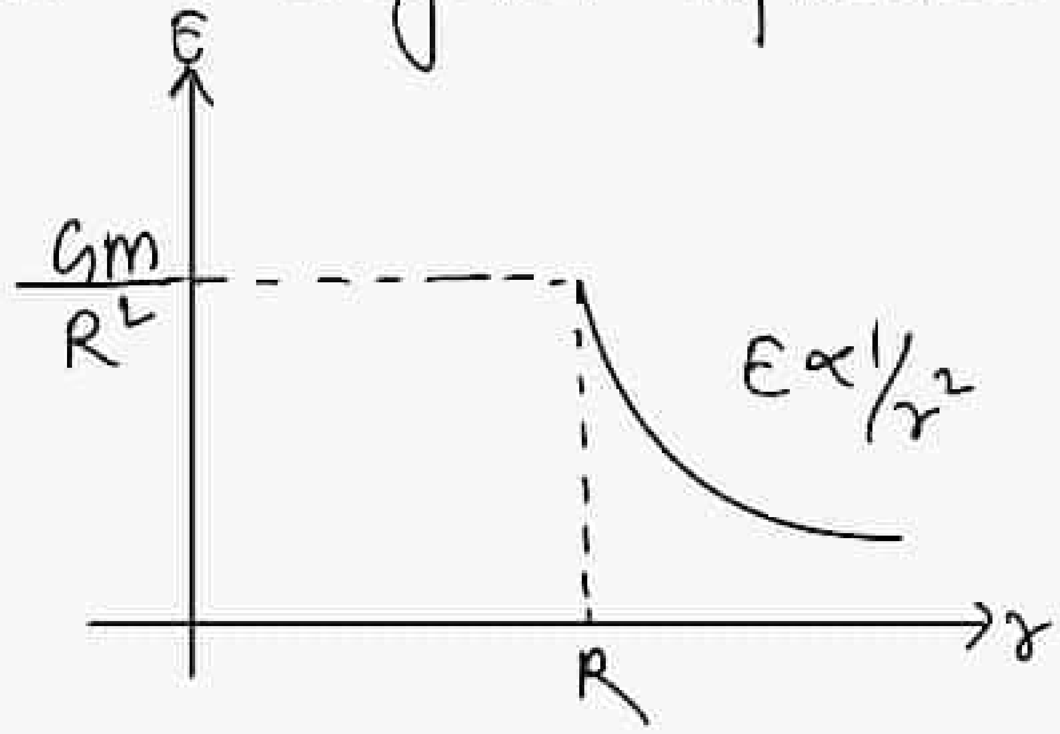
$$\boxed{E(r) = \frac{GM}{r^2}} \quad \text{for } r \geq R$$

at $r = R$ (the surface of shell)

$$E = \frac{GM}{R^2} \quad \text{and otherwise } E \propto 1/r^2$$

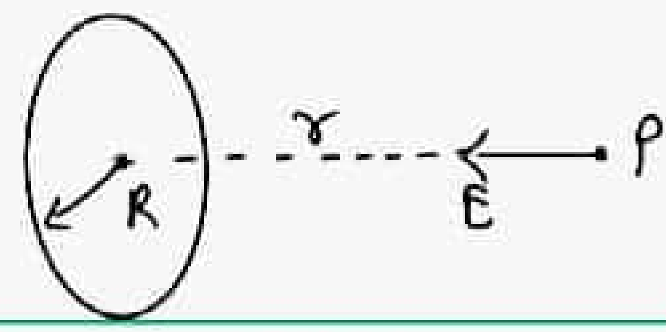
⇒ At an Internal Point :-

The field inside a uniform spherical shell is zero.



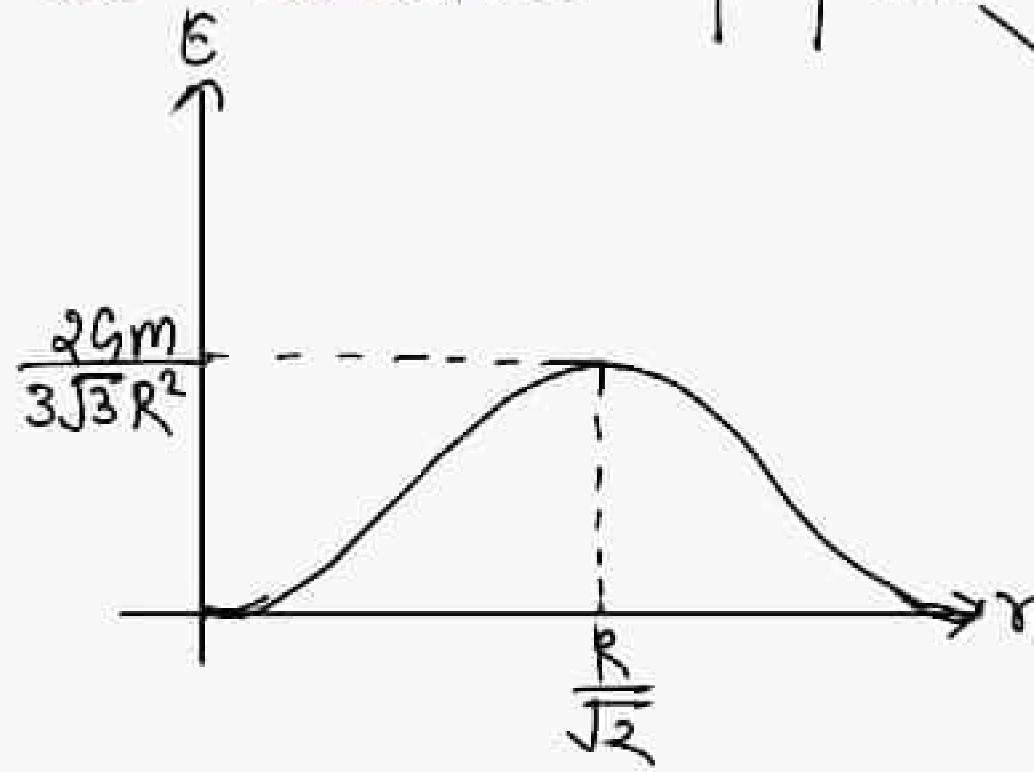
⇒ Field due to a Uniform Circular Ring at some Point on its Axis :-

Field Strength at a point P on the axis of a circular ring of radius R and mass M is given by,



$$\boxed{E(r) = \frac{GMr}{(R^2+r^2)^{3/2}}}$$

This is directed towards the centre of the ring. It is zero at the centre of the ring and maximum at $r = \frac{R}{\sqrt{2}}$ (can be obtained by putting $\frac{dE}{dr} = 0$) Thus, E-r graph is



The maximum value is

$$E_{max} = \frac{2GM}{3\sqrt{3}\,R^2}$$

Gravitational Potential :-

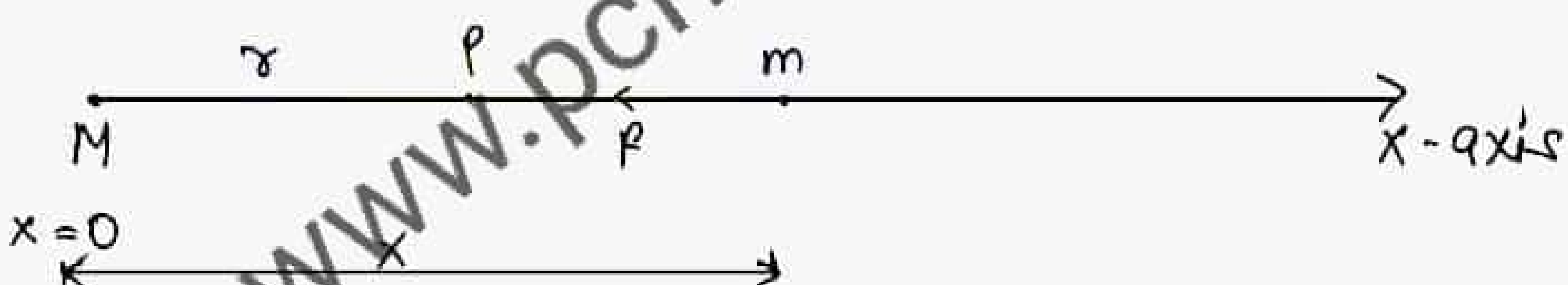
Gravitational potential at any point is defined as the negative of work done by gravitational force in moving a unit test mass from infinity (where potential is assumed to be zero) to that point.

Thus, potential at P is

$$\boxed{V_P = \frac{-W_{\infty \to P}}{m}}$$ (by gravitational force)

It's SI Unit is J/kg

⇒ Potential due to a Point mass :-



$$\boxed{F = -\frac{GMm}{x^2}}$$

$$W = \int_{\infty}^{r} F\,dx = \int_{\infty}^{r} \left(-\frac{GMm}{x^2}\right) dx = \frac{GMm}{r}$$

Now, from the definition of potential,

$$V = -\frac{W}{m} = -\frac{GM}{r}$$

$$\therefore \quad \boxed{V = -\frac{GM}{r}}$$

Potential Due to a Uniform Solid Sphere:-

⇒ Potential at some External Point

The gravitational potential due to a uniform sphere at an external point is same as that due to a single particle of same mass placed at its centre. Thus,

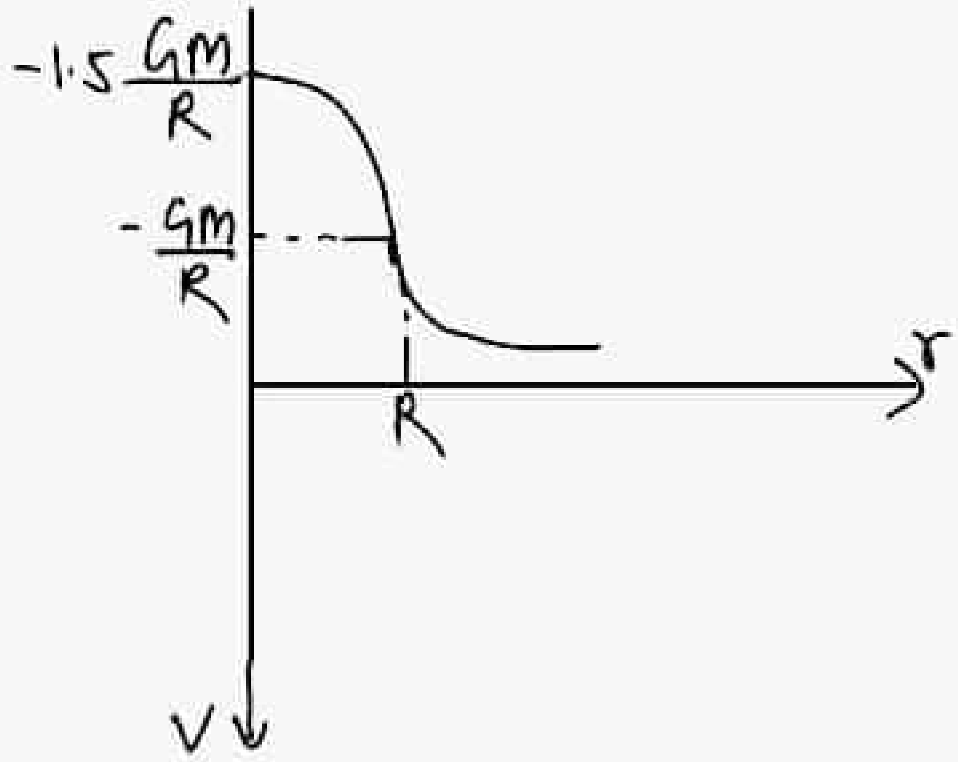
$$\boxed{V(r) = -\frac{GM}{r}} \quad r \geq R$$

At the surface, $r = R$ and $V = -\frac{GM}{R}$

⇒ Potential at some Internal Point :-

At some internal point, potential at a distance r from the centre is given by,

$$\boxed{V(r) = \frac{-GM}{R^3}\left(1.5R^2 - 0.5r^2\right)} \quad r \leq R$$



At $r=R$, $V = \frac{-GM}{R}$

while at $r=0$, $V = \frac{-1.5GM}{R}$

i.e. at the centre of the sphere the potential is 1.5 times the potential at the surface. The variation of V versus r graph is as shown in figure

Potential due to a Uniform Thin Spherical Shell:-

⇒ Potential at an External Point :-

To calculate the potential at an external point, a uniform spherical shell may be treated as a point mass of same magnitude at its centre.

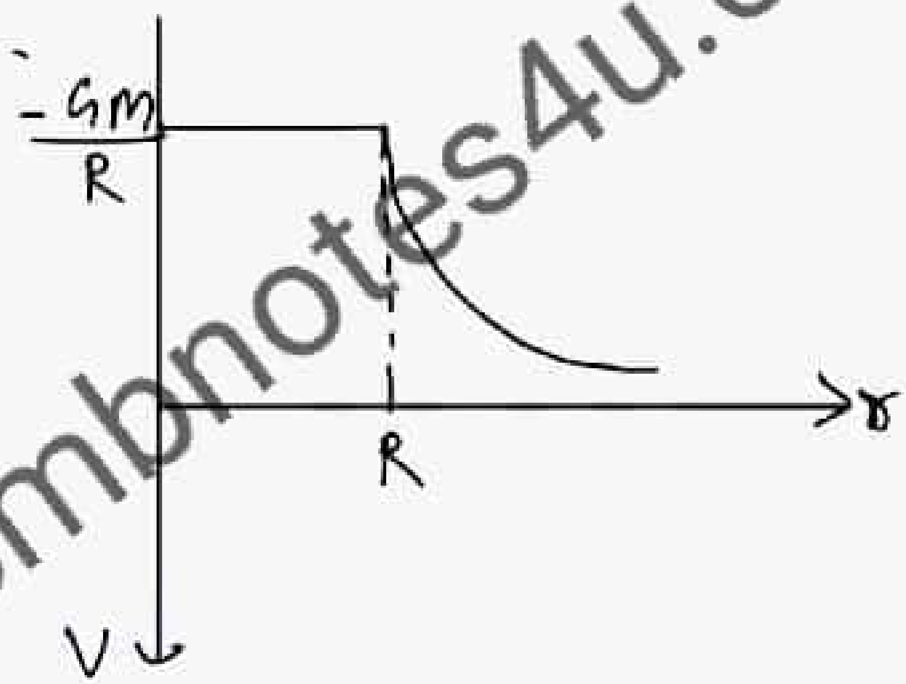
Thus, potential at a distance r is given by,

$$\boxed{V(r) = \frac{-GM}{r}} \quad r \geq R$$

at $r = R$, $V = -\dfrac{GM}{R}$

⇒ Potential at an Internal Point :-

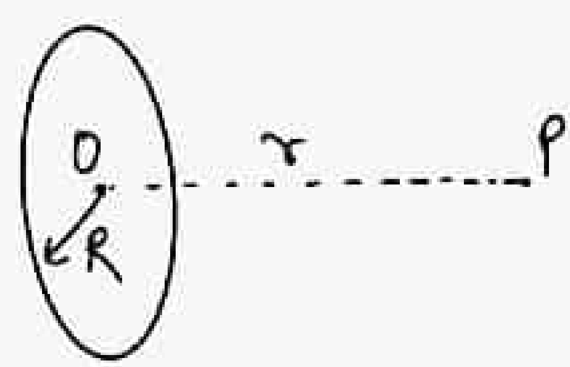
The potential due to a uniform spherical shell is constant at any point inside the shell and this is equal to $-\dfrac{GM}{R}$. Thus, V-r graph for a spherical shell is as shown :-



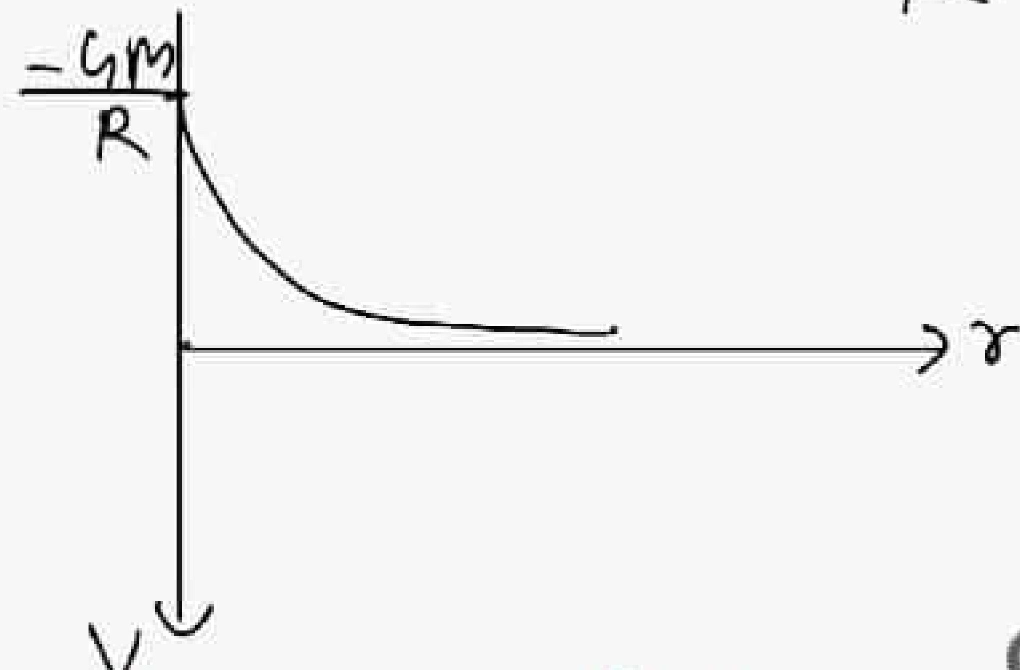
⇒ Potential due to a Uniform Ring at some Point on its Axis :-

The gravitational potential at a distance r from the centre on the axis of a ring of mass M and radius R is given by,

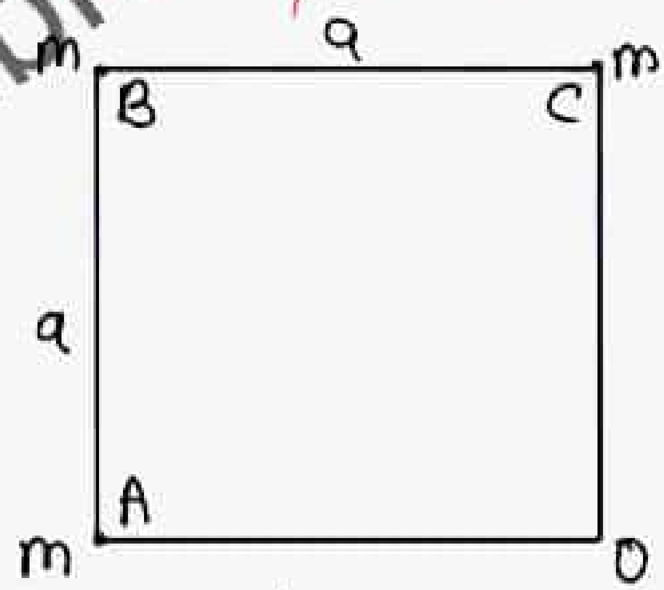
$$\boxed{V(r) = -\frac{GM}{\sqrt{R^2 + r^2}}} \quad 0 \le r \le \infty$$



At $r = 0$, $V = -\frac{GM}{R}$, i.e. at the centre of the ring gravitational potential is $-\frac{GM}{R}$.



Eg:- Three point masses 'm' each are kept at three vertices of a square of side 'a' as shown in figure. Find gravitational potential and field strength at point O.



Solⁿ:- Gravitational potential is a scalar quantity.

$\therefore$ V_O = scalar sum of potentials due to point masses at A, B and C.

$$= -\frac{GM}{a} - \frac{GM}{\sqrt{2}a} - \frac{GM}{a}$$

$$= -\frac{GM}{a}\left[2 + \frac{1}{\sqrt{2}}\right] \quad \text{— (Ans)}$$

Relation between Gravitational field and Potential :-

$$E = -\text{gradient } V = -\left[\frac{\partial V}{\partial x}\hat{i} + \frac{\partial V}{\partial y}\hat{j} + \frac{\partial V}{\partial z}\hat{k}\right]$$

or $$\vec{E} = -\left[\frac{\partial V}{\partial x}\hat{i} + \frac{\partial V}{\partial y}\hat{j} + \frac{\partial V}{\partial z}\hat{k}\right]$$

Here, $\frac{\partial V}{\partial x}$ is called partial differentiation of V with respect to x

⇒ Only one Variable :-

$$\boxed{E = -\frac{dv}{dx} = -\frac{dv}{dr}}$$ or $E = (-\text{slope of } V\text{-}x)$ or $(-\text{slope of } V\text{-}r \text{ graph})$

Conversion of E function into V function:-

⇒ More than one Variables:-

In this case, $\boxed{dV = E \cdot dr}$

or $$\int_a^b dV = -\int_a^b E \cdot dr$$

$$\Rightarrow \quad V_b - V_a = -\int_a^b E \cdot dr$$

Here, E will be given in the question and dr is a standard vector given by

$$dr = dx\hat{i} + dy\hat{j} + dz\hat{k}$$

⇒ Only one Variable

$$\boxed{\int dV = -\int E\,dr}$$

or $$\int_a^b dV = -\int_a^b E\,dr$$

$$\Rightarrow \quad V_b - V_a = -\int_a^b E\,dr$$

Here, E as a function of r will be given in the question. We may also write the above eq^n as

$$\int dV = -\int E\,dx$$

where, E is a function of x.

Eg:- Gravitational potential in x-y plane varies with x and y coordinates as

$$V = x^2y + 2xy$$

Find gravitational field strength E.

Sol^n:- Gravitational field strength is given by

$$\vec{E} = -\left[\frac{\partial v}{\partial x}\hat{i} + \frac{\partial v}{\partial x}\hat{j}\right]$$

for the given potential function

$$\frac{\partial v}{\partial x} = (2xy + 2y)$$

and $$\frac{\partial v}{\partial y} = x^2 + 2x$$

$$\therefore \quad \vec{E} = -\left[(2xy + 2y)\hat{i} + (x^2 + 2x)\hat{j}\right] \rightarrow (\text{Ans})$$

Eg:- Gravitational potential at a distance 'r' from a point mass 'm' is

$$V = \frac{-GM}{r}$$

Find gravitational field strength at that point.

Sol^n:- $$\vec{E} = -\frac{dv}{dr}$$

$$= -\frac{d}{dr}\left(-\frac{Gm}{r}\right) = -\frac{Gm}{r^2} \quad \text{or} \quad |E| = \frac{Gm}{r^2}$$

Gravitational Potential Energy :-

$$dU = -F \cdot dr \quad \text{or} \quad \int_i^f dU = -\int_{r_i}^{r_f} F \cdot dr \quad \text{or} \quad U_f - U_i = -\int_{r_i}^{r_f} F \cdot dr$$

$$\boxed{U = -\int_{\infty}^{r} F \cdot dr = -W}$$

⇒ Gravitational Potential Energy of a two Particle System:-

The gravitational potential energy of two particles of masses m_1 and m_2 separated by a distance r is given by,

m_1 - - - - - - r - - - - - - m_2

$$\boxed{U = -\frac{Gm_1m_2}{r}}$$

This is actually the negative of work done in bringing those masses from infinity to a distance r by the gravitational forces between them.

⇒ Gravitational Potential Energy for a System of Particles:-

The gravitational potential energy for a system of particles (say m_1, m_2, m_3 and m_4) is given by-

$$U = -G\left[\frac{m_4 m_3}{r_{43}} + \frac{m_4 m_2}{r_{42}} + \frac{m_4 m_1}{r_{41}} + \frac{m_3 m_2}{r_{32}} + \frac{m_3 m_1}{r_{31}} + \frac{m_2 m_1}{r_{21}}\right]$$

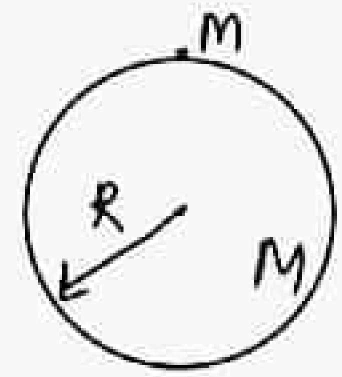
Thus, for a n particle system there are $\frac{n(n-1)}{2}$ pairs and the potential energy is calculated for each pair and added to get the total potential energy of the system.

⇒ Gravitational Potential Energy of a Body on Earth's Surface:-

The gravitational potential energy of mass m in the gravitational field of mass M at a distance r from it is,

$$\boxed{U = -\frac{GMm}{r}}$$

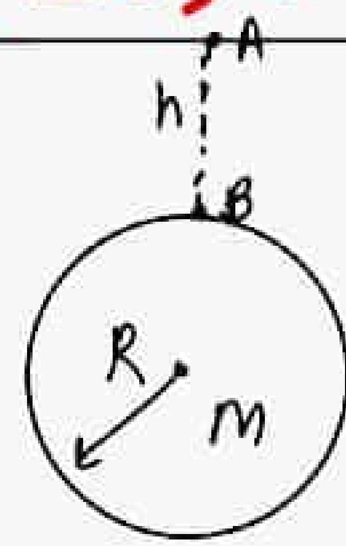
The earth behaves for all external points as if its mass M were concentrated at its centre. Therefore, a mass m near earth's surface may be considered at a distance R (the radius of earth) from M. Thus, the potential energy of the system will be



$$U = -\frac{GMm}{R}$$

Difference in Potential Energy (ΔU):-

Let us find the difference in potential energy in two positions shown in fig.



The potential energy when the mass is on the surface of earth (at B) is,

$$U_B = -\frac{GMm}{R}$$

and potential energy when the mass m is at height h above the surface of earth (at A) is,

$$U_A = -\frac{GMm}{R+h} \qquad (U_A > U_B)$$

$$\therefore \quad \Delta U = U_A - U_B$$

$$= -\frac{GMm}{R+h} - \left(-\frac{GMm}{R}\right)$$

$$= GMm\left(\frac{1}{R} - \frac{1}{R+h}\right)$$

$$= \frac{GMmh}{R(R+h)}$$

$$= \frac{GMmh}{R^2\left(1+\frac{h}{R}\right)} = \frac{mgh}{\left(1+\frac{h}{R}\right)} \quad \left(\frac{GM}{R^2} = g\right)$$

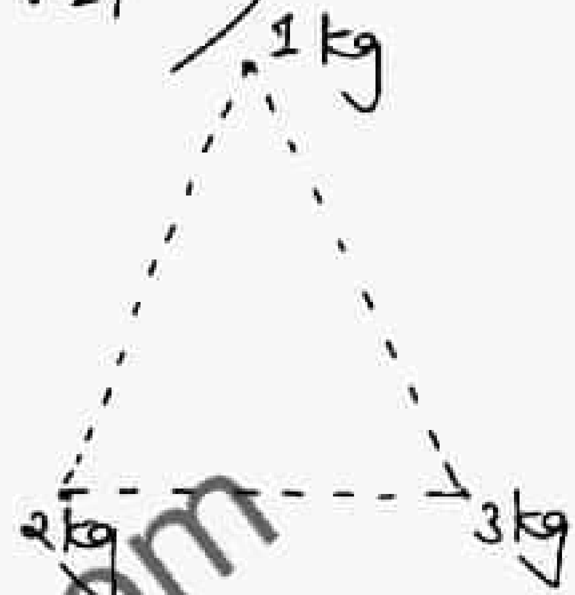
$$\therefore \boxed{\begin{array}{c} \Delta U = \dfrac{mgh}{1+\frac{h}{R}} \\ h << R, \ \Delta U \approx mgh \end{array}}$$

Thus, mgh is the difference in potential energy (not the absolute potential energy), for $h << R$.

Eg:- Three masses of 1 kg, 2 kg and 3 kg are placed at the vertices of an equilateral triangle of side 1 m. Find the gravitational potential energy of this system.

Take $G = 6.67 \times 10^{-11}\ N\text{-}m^2/kg^2$

Sol^n :- $U = -G\left(\dfrac{m_3 m_2}{r_{32}} + \dfrac{m_3 m_1}{r_{31}} + \dfrac{m_2 m_1}{r_{21}}\right)$



Here, $r_{32} = r_{31} = r_{21} = 1.0\ m$,

$m_1 = 1\ kg$,

$m_2 = 2\ kg$,

and $m_3 = 3\ kg$

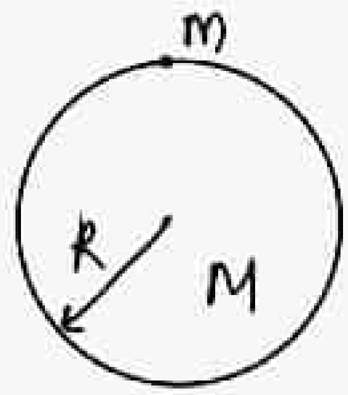
Substituting the values, we get

$$U = -(6.67 \times 10^{-11})\left(\frac{3 \times 2}{1} + \frac{3 \times 1}{1} + \frac{2 \times 1}{1}\right)$$

or $\boxed{U = -7.337 \times 10^{-10}\ J}$ $\Rightarrow$ (Ans)

Binding Energy :-

Total mechanical energy (potential + kinetic) of a closed system is negative. The modulus of this total mechanical energy is known as the binding energy of the system. This is the energy due to which system is closed or different parts of the system are bound to each other.



Suppose, the mass m is placed on the surface of earth. The radius of the earth is R and its mass is M. Then, the kinetic energy of the particle $k=0$ and the potential energy is

$$\boxed{U = -\frac{GMm}{R}}$$

Therefore, the total mechanical energy is,

$$E = K + U = 0 - \frac{GMm}{R}$$

or $$E = -\frac{GMm}{R}$$

$\therefore$ Binding energy $= |E| = \frac{GMm}{R}$

It is due to this energy, the particle is attached with the earth. If minimum this much energy is given to the particle in any form (normally kinetic) the particle no longer remains attached to the earth. It goes out of the gravitational field of earth.

Escape Velocity:-

As we discussed above, the binding energy of a particle on the surface of earth kept at rest is $\frac{GMm}{R}$. If this much energy in the form of kinetic energy is supplied to the particle, it leaves the gravitational field of the earth. So, if Ve is the escape velocity of the particle, then

$$\frac{1}{2} m v_e^2 = \frac{GMm}{R} \quad \text{or} \quad V_e = \sqrt{\frac{2GM}{R}}$$

$$\text{or} \quad \boxed{V_e = \sqrt{2gR}} \quad \text{as} \quad g = \frac{GM}{R^2}$$

Substituting the value of g ($9.8\, m/s^2$) and R ($6.4 \times 10^6\, m$), we get

$$\boxed{V_e \approx 11.2\ km/s}$$

Thus, the minimum velocity needed to take a particle to infinity from the earth is called the escape velocity. On the surface of earth its value is 11.2 km/s.

Eg:- Calculate the escape velocity from the surface of moon. The mass of the moon is 7.4×10^{22} kg and radius = 1.74×10^{6} m.

Sol^n:- Escape velocity from the surface of moon is $V_e = \sqrt{\dfrac{2GM_m}{R_m}}$

Substituting the values, we have

$$V_e = \sqrt{\frac{2 \times 6.67 \times 10^{-11} \times 7.4 \times 10^{22}}{1.74 \times 10^{6}}}$$

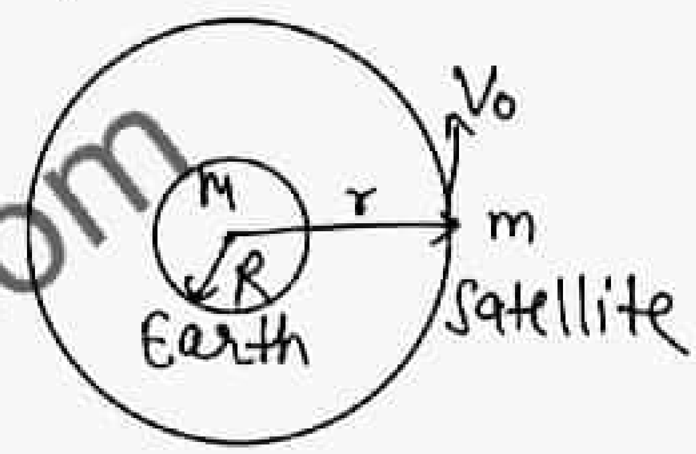
$= 2.4 \times 10^{3}$ m/s $= 2.4$ Km/s $\Longrightarrow$ (Ans)

Motion of Satellites :-

⇒ Orbital Speed :-

The necessary centripetal force to the satellite is being provided by the gravitational force exerted by the earth on the satellite. Thus,

$$\frac{m v_o^2}{r} = \frac{GMm}{r^2}$$



$$\therefore \quad \boxed{v_o = \sqrt{\frac{GM}{r}}} \quad \text{or} \quad v_o \propto \frac{1}{\sqrt{r}}$$

Hence, the orbital speed (v_o) of the satellite decreases as the orbital radius (r) of the satellite increases. Further, the orbital speed of a satellite close to the earth's surface ($r \approx R$) is,

$$\boxed{v_o = \sqrt{\frac{GM}{R}} = \sqrt{gR} = \frac{v_e}{\sqrt{2}}}$$

Substituting,

$$v_e = 11.2 \text{ km/s} \Rightarrow \boxed{v_o = 7.9 \text{ km/s}}$$

⇒ Period of Revolution :-

The period of revolution (T) is given by

$$T = \frac{2\pi r}{v_0} \quad \text{or} \quad T = \frac{2\pi r}{\sqrt{\frac{GM}{r}}}$$

$$\text{or} \quad T = 2\pi\sqrt{\frac{r^3}{GM}} \quad \text{or} \quad \boxed{T = 2\pi\sqrt{\frac{r^3}{gR^2}}} \quad (\text{as } GM = gR^2)$$

⇒ Energy of Satellite :-

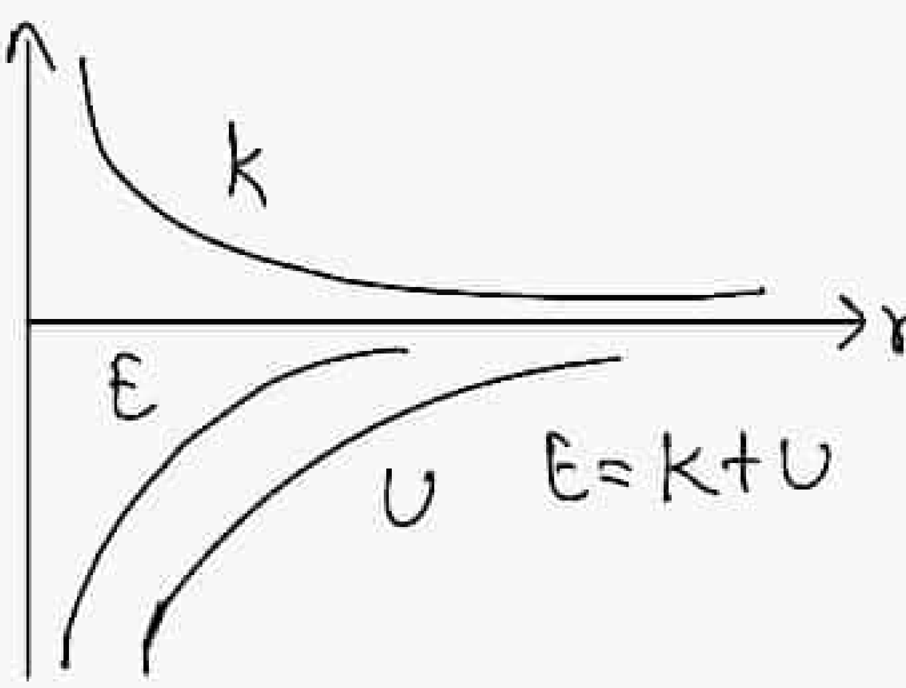
The potential energy of the system is

$$\boxed{U = -\frac{GMm}{r}}$$

The kinetic energy of the satellite is,

$$K = \frac{1}{2} m v_0^2 = \frac{1}{2} m \left(\frac{GM}{r}\right)$$

$$\text{or} \quad \boxed{K = \frac{1}{2}\frac{GMm}{r}}$$



The total energy is, $E = K + U = \frac{-GMm}{2r}$

or
$$\boxed{E = \frac{-GMm}{2r}}$$

This energy is constant and negative, i.e. the system is closed. The farther the satellite from the earth the greater its total energy.

Points to Remember :-

① $T = 2\pi\sqrt{\frac{r^3}{gR^2}} \Rightarrow T \propto r^{3/2}$ or $T^2 \propto r^3$ (which is also the Kepler's third law)

② Time period of a satellite very close to earth's surface ($r \approx R$) is,

$$T = 2\pi\sqrt{\frac{R}{g}}$$

Substituting the values, we get $T \approx 84.6$ min

Eg:- As orbital radius r of a satellite is increased, state which of the following quantities will increase and which will decrease?

(i) Orbital speed (ii) Time Period
(iii) Frequency (iv) Angular Speed
(v) Kinetic energy (vi) Potential energy
(vii) Total mechanical energy

Solⁿ:- (i) Orbital Speed $V_o = \sqrt{\frac{GM}{r}}$ or $V_o \propto \frac{1}{\sqrt{r}}$

Therefore, orbital speed will decrease.

(ii) $T = 2\pi \sqrt{\frac{r^3}{gR^2}}$ or $T \propto r^{3/2}$

Therefore, time period will increase.

(iii) Frequency, $f = 1/T$

Time period is increasing. So, frequency will decrease.

(iv) Angular Speed $\omega = \frac{2\pi}{T}$ or $\omega \propto \frac{1}{T}$

Time period is increasing. Hence, angular speed will decrease.

(v) Kinetic Energy, $K = \frac{GMm}{2r}$ or $K \propto \frac{1}{r}$

Therefore, kinetic energy will decrease.

(vi) Potential Energy, $U = \frac{-GMm}{r}$ or $U \propto -\frac{1}{r}$

Therefore, potential energy will increase.

(vii) Total Mechanical Energy, $E = -\frac{GMm}{2r}$ or $E \propto -\frac{1}{r}$

So, mechanical energy will also increase

Eg:- What is the minimum energy required to launch a satellite of mass m from the surface of a planet of mass M and radius R in a circular orbit at an altitude of 2R?

(a) $\frac{5GmM}{6R}$ (b) $\frac{2GmM}{3R}$

(c) $\frac{GmM}{2R}$ (d) $\frac{GmM}{3R}$

Solⁿ:- E = Energy of Satellite − energy of mass on the surface of planet

$$= -\frac{GMm}{2r} - \left(-\frac{GMm}{R}\right)$$

Here, $r = R + 2R = 3R$

Substituting the values in above eqⁿ we get,

$$E = \frac{5GMm}{6R}$$

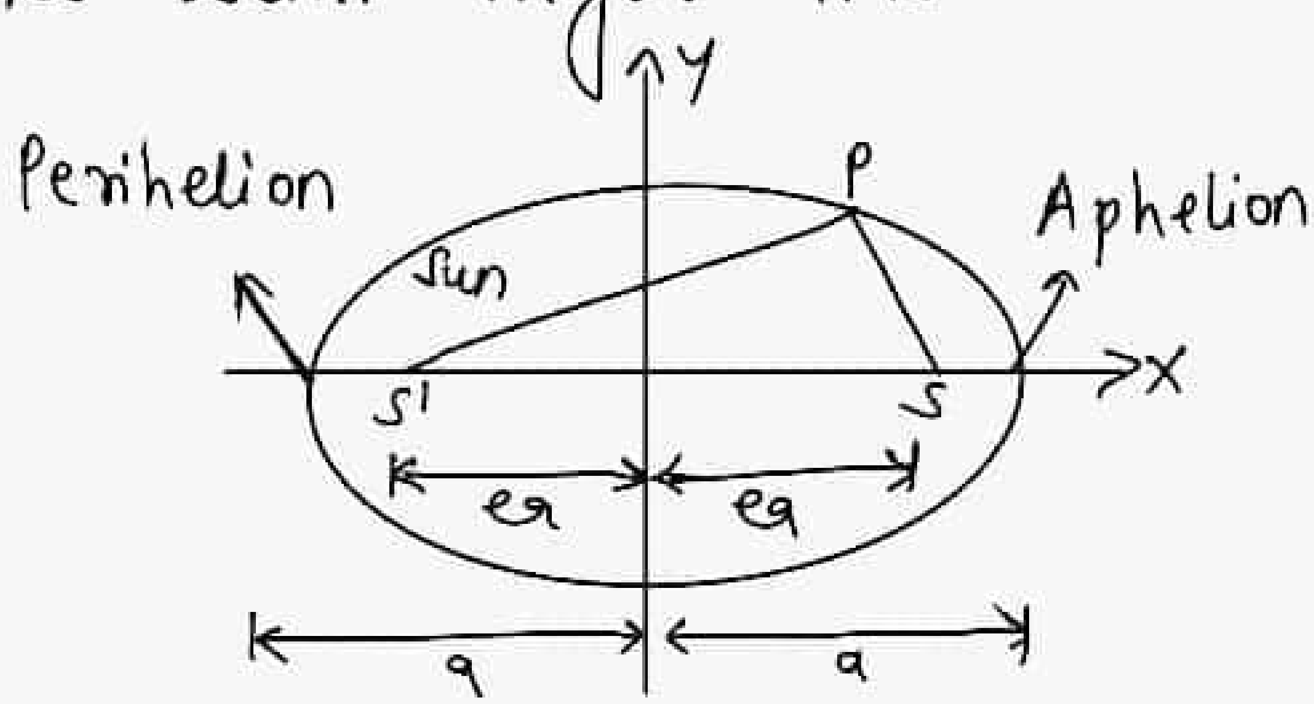
The correct answer is (a)

Kepler's laws of Planetary Motion:-

(1) Each planet moves in an elliptical orbit, with the sun at one focus of the ellipse. This law is also known as the law of elliptical orbits and obviously gives the shape of the orbits of the planets round the sun.

(2) The radius vector, drawn from the sun to a planet, sweeps out equal areas in equal time, i.e. its areal velocity (or the area swept out by it per unit time) is constant. This is referred to as the law of areas and gives the relationship b/w the orbital speed of the planet and its distance from the sun.

(3) The sq. of the planet's time period is proportional to the cube of the semi-major axis of its orbit. This is known as the law of time period and gives the relationship b/w the size of the orbit of a planet and its time of revolution. The longest dimension is the major axis with half length a. This half length is called the semi-major axis.



Here, S and S' are the foci and P any point on the ellipse. The sun is at S and planet at P. The distance of each focus from the centre of ellipse is ea, where e is the dimensionless no. between 0 to 1 called the eccentricity. If e=0, the ellipse is a circle. The actual orbits of the planets are nearly circular, their eccentricities range from 0.007 for Venus to 0.248 for Pluto. for earth e=0.017. The point in the planet's orbits closest to the sun is the perihelion and the point most distant from the sun is aphelion.

Explanation of First Law:-

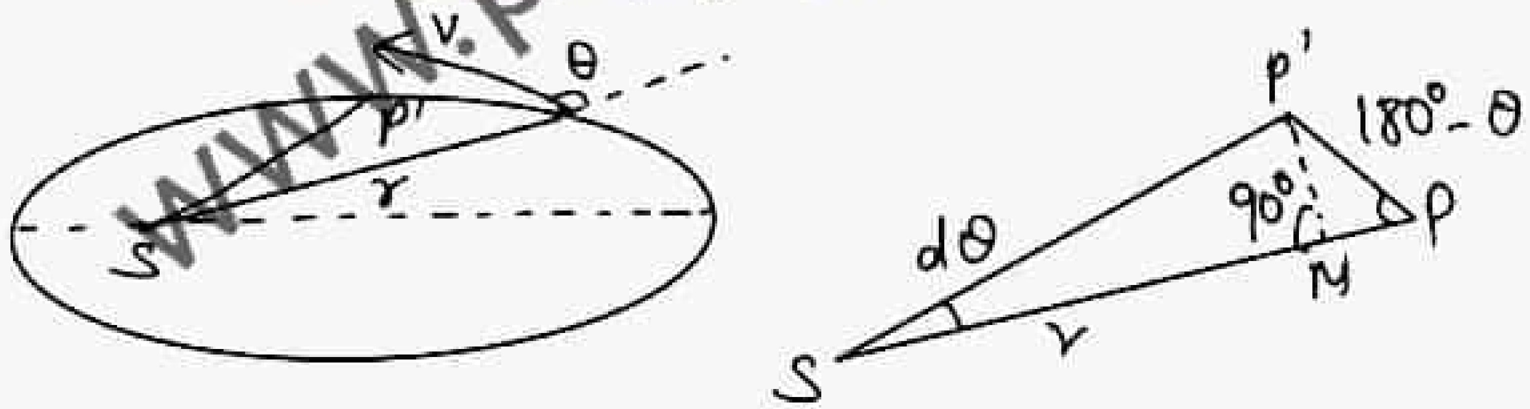
Newton was able to show that for a body acted on by an attractive force proportional to $1/r^2$, the only possible closed orbits are a circle or an ellipse. The open orbits must be parabolas or hyperbolas. He also showed that if total energy E is negative the orbit is an ellipse (or circle), if it is zero the Orbit is a parabola and if E is positive the orbit is a hyperbola. further, it was also shown that the orbits under the attractive force $F = \frac{k}{r^n}$, are stable for $n < 3$. Therefore, it follows that circular orbits will be stable for a force varying inversely as the distance or the sq. of the distance and will be unstable for the inverse cube (or a higher power) law.

Explanation of Second law:-

$$PP' = v\,dt$$

$$P'M = (PP')\sin(180^\circ - \theta) = PP'\sin\theta$$

$$= (v\sin\theta)\,dt$$



Keepler's 2^{nd} law is shown in fig. In a small time interval dt, the line from the sun S to the planet P turns through an angle $d\theta$. The area swept out in this time interval is,

dA = area of triangle shown in fig.

$$= \frac{1}{2} \text{(base) (height)}$$

$$= \frac{1}{2} (SP)(P'M)$$

$$= \frac{1}{2} (r)(v \sin\theta)\, dt$$

$\therefore$ Areal Velocity $\frac{dA}{dt} = \frac{1}{2} rv \sin\theta$ —— (1)

Now, $rv \sin\theta$ is the magnitude of the vector product $r \times V$ which in turn is $\frac{1}{m}$ times the angular momentum $L = r \times mv$ of the planet with respect to the sun. So, we have,

$$\frac{dA}{dt} = \frac{1}{2m} |r \times mv| = \frac{L}{2m} \quad \text{—— (2)}$$

or $$\boxed{\frac{dA}{dt} = \frac{L}{2m} = \text{Constant}}$$

Thus, Kepler's second law, that areal velocity is constant, means that angular momentum is constant. It is easy to see why the angular momentum of the planet must be constant. Acc. to Newton's law the rate of change of L equals the torque of the gravitational force F acting on the planet,

$$\frac{dL}{dt} = r \times f = \tau$$

Here, r is the radius vector of planet from the centre of the sun and the force F is directed from the planet towards the centre of the sun. So, these vectors always lie along the same line and their vector product $r \times F$ is zero. Hence, $\frac{dL}{dt} = 0$ or L = constant. Thus, from eq^n (2), we can see that $\frac{dA}{dt}$ = constant if L = constant.

Thus, 2nd law is actually the law of conservation of angular momentum

Explanation of Third law :-

We have already derived kepler's third law for the particular case of circular orbits ($T^2 \propto r^3$). Newton was able to show that the same relationship holds for an elliptical orbit, with the orbit radius r replaced by semimajor axis a.

Thus,

$$\boxed{T = \frac{2\pi a^{3/2}}{\sqrt{GM_s}}} \qquad \text{(elliptical orbit)}$$

Here, M_s is the mass of the Sun.